Uday Khankhoje

Antenna arrays & Fourier theory

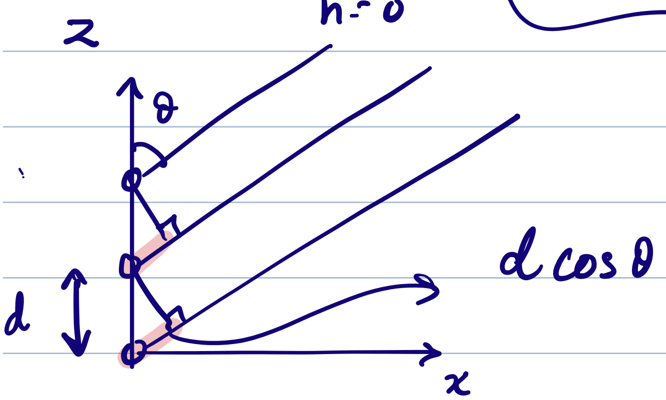
We now explore the connections between antenna array theory & signal processing.

① Start with the array factor (AF) for a uniformly spaced 1D array (spacing d, wavevector k):

We know $E_{total}(\theta) = f(\theta)\, AF(\theta)$ (by pattern multiplication)

$$AF(\theta) = \sum_{n=0}^{N-1} a_n e^{jkd\cos\theta n}$$

complex array excitation (mag) and (phase)



Set $\psi = kd\cos\theta$

$$AF(\psi) = \sum_{n=0}^{N-1} a_n e^{jn\psi}.$$

② Let's analyse $AF(\psi)$. Is it periodic?

$AF(\psi + 2\pi) = AF(\psi)$. Yes, with period 2π, for any $\{a_n\}$. $\Rightarrow$ We only need to know AF over one period, e.g. $[-\pi, \pi]$.

③ A common statement is "the AF is just a Fourier transform". Is it true?

Consider 3 transforms:

Transform	Defn	Argument
1) DTFT	$A(\psi) = \sum_{n=0}^{N-1} a_n e^{jn\psi}$	ψ conts.
2) DFT	$A[m] = \sum_{n=0}^{N-1} a_n e^{j\frac{2\pi}{N}nm}$	m discrete $m \in [0, N-1]$
3) z-transform	$A(z) = \sum_{n=0}^{N-1} a_n z^{-n}$	$z \in \mathbb{C}$

aside

Reminder of ①: $A(\omega) \triangleq \int_{-\infty}^{\infty} a(t) e^{-j\omega t} dt$

Sample it every T:

$$A(\omega)\Big|_{1/T} = \sum_{-\infty}^{\infty} \underbrace{a(nT)T}_{a_n} e^{-j\omega nT}$$

$$= \sum_{-\infty}^{\infty} a_n e^{-jn(\omega T)}$$

So, what is our AF? DTFT of $\{a_n\}$ evaluated at a contns frequency ψ.

Then if we sample the DTFT at discrete freqs $\psi = \frac{2\pi m}{N}$, we get the DFT of $\{a_n\}$

i.e. $A[m] = AF\left(\psi = \frac{2\pi m}{N}\right)$.

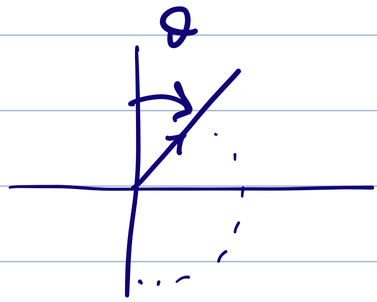
So if we do AF_samples = fft(a, M)
what are we doing? (M log M)

1) M − point FFT
$\Rightarrow$ M uniform samples in $[0, 2\pi)$
i.e. $\psi = \left[0, \frac{2\pi}{N}, \frac{2\pi}{N}\cdot 2 \cdots, \frac{2\pi}{N}(M-1)\right]$

2) If $M > N$?
$\{a_n\}$ is zero padded to length M

Summary: The AF is technically a DTFT, when we take a FFT of $\{a_n\}$ we are sampling the DTFT.

④ The visible region.



As $\theta \to 0$ to π,
$u = \cos\theta$ goes
$u \to 1$ to -1
and $\psi = kd\cos\theta$ goes
$\psi \to kd$ to $-kd$
i.e. $\theta \to [0, \pi]$ maps ψ $[-kd, kd]$

In ψ space, we are spanning $kd + kd$
$= 2kd \rightarrow$ defined as the visible region since it corresponds to $\theta = 0$ to π (physically visible)

[Recall this is a 1D problem, symmetric about z axis, so no need for $\theta > \pi$]

Now: $2kd = 2 \cdot \frac{2\pi}{\lambda} \cdot d = \frac{4\pi d}{\lambda}$.

and the periodicity of AF is 2π.

↳ ∴ At $d = \lambda/2$, $2kd = 2\pi$

⇒ Every feature of AF(ψ) is seen once.

↳ If $d < \lambda/2$? $2kd < 2\pi$

⇒ As θ goes from $0 \to \pi$, AF plot covers less than 2π.

⇒ Some features of AF are hidden.

↳ If $d > \lambda/2$? $2kd > 2\pi$

⇒ As θ goes from $0 \to \pi$, AF plot covers more than 2π.

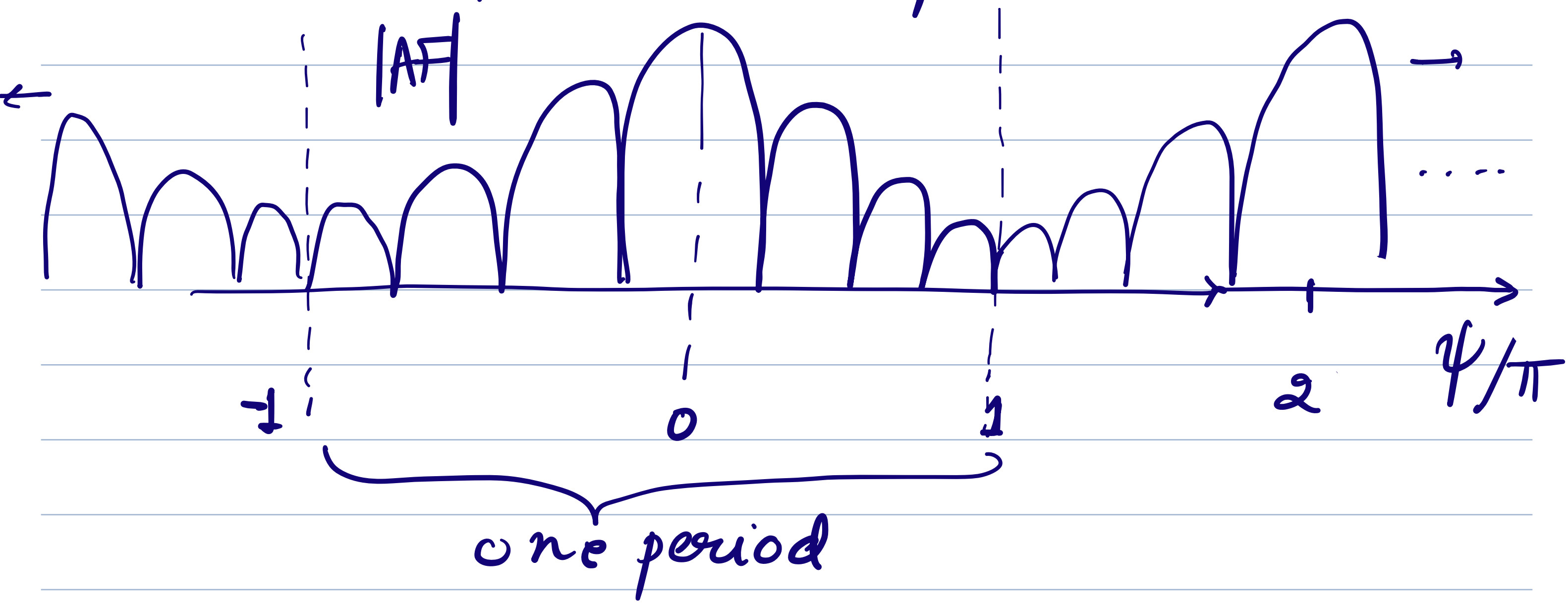
→ But AF is still periodic with 2π.

⇒ Parts of adjacent periods enter the visible region → Grating lobes (an alternate explanation).

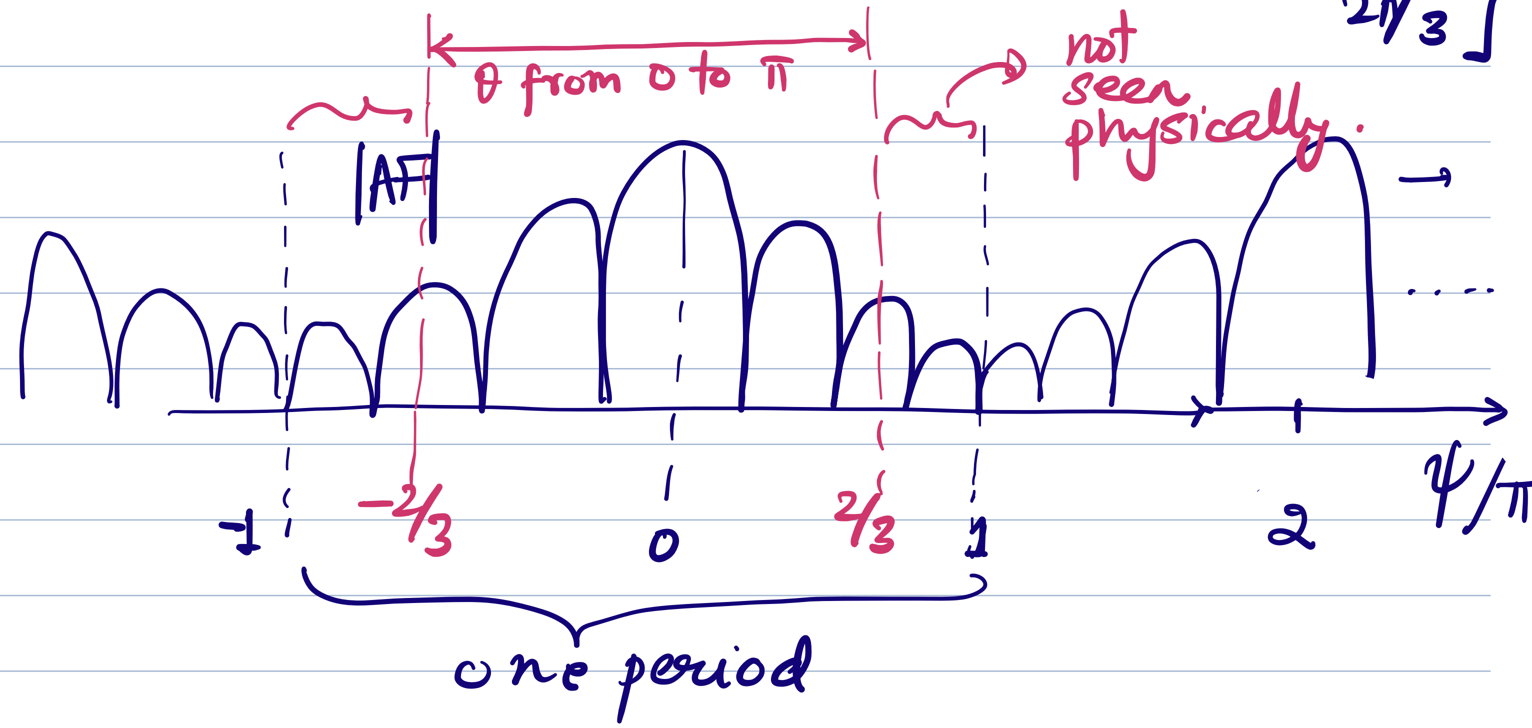
Numerical example with $N = 8$, $a_n = 1 \ \forall n$.

$$AF = \frac{\sin N\psi/2}{\sin \psi/2} \quad , \quad \psi = kd\cos\theta$$

$\therefore$ AF v/s ψ doesn't depend on kd.

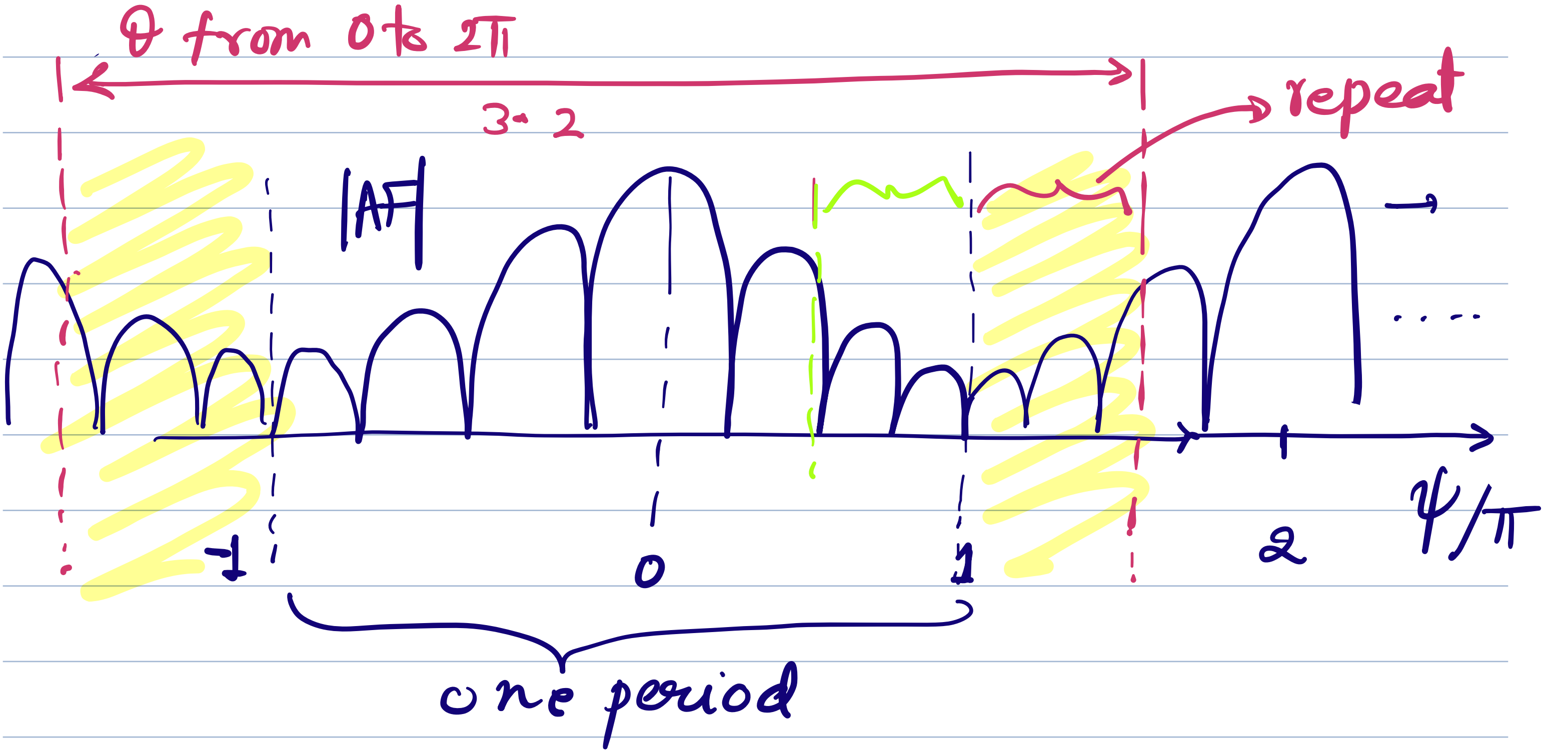


↳ case 1, $d = \lambda/3 \Rightarrow kd = 2\pi/3$, visible $\psi \in [-2\pi/3, 2\pi/3]$



↳ case 2, $d = 0.8\lambda \Rightarrow kd = 1.6\pi$

$\Rightarrow$ visible region $\psi \in [-1.6\pi, 1.6\pi]$

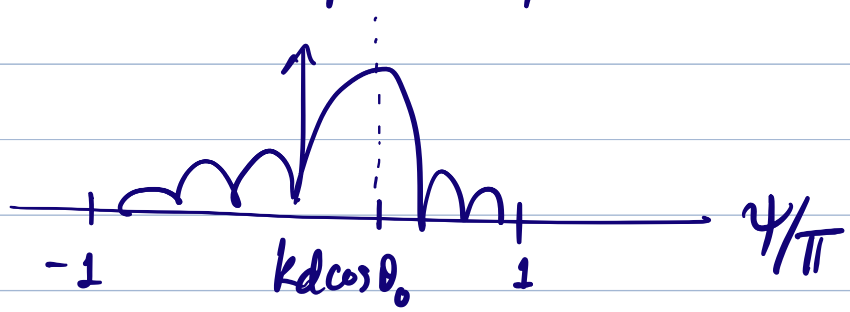


The shaded regions are being seen twice when θ goes from 0 to π.

When $d = 1\lambda$, $kd = 4\pi \Rightarrow$ we see the main lobe from other periods enter into the visible region.
Mostly undesirable!

<u>A note</u>: if progessive phase shift, then $AF = \sum a_n e^{jn\psi'}$ with $\psi' = kd\cos\theta + \beta$. If we want beam to form at $\theta = \theta_0$, then $AF = \dfrac{\sin N\psi'/2}{\sin \psi'/2} \Rightarrow kd\cos\theta_0 + \beta = 0$.

with $\psi' = \psi + \beta$, the $AF(\psi)$ plot will have a maxima at $\psi = -\beta. = kd\cos\theta_0$



↳ In summary, analogy with the sampling theorem.

Signal $f(t)$, conts time, with b/w B → sampled every $t = T_s$ $\left(f_s = \frac{1}{T_s}\right)$

↓

Nyquist condition, $f_s > 2B$
else, overlapping (aliasing)

In the antenna array case?

Aperture distribution $a(z)$ (continuous z) → sampled every $z = d$, i.e $f_s = \frac{1}{d}$.

↓

Spatial Nyquist condn
if $f_s = \frac{1}{d} > \frac{2}{\lambda}$ $\left(\text{i.e } d < \frac{\lambda}{2}\right)$

else, grating lobes

⇒ We can think of the signal $a(z)$ having spatial bandwidth $= \frac{1}{\lambda}$

(by analogy with B above)