Theoretical & Computational Electromagnetics

Parity-Time Symmetry in Electromagnetics

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Outline

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1. Section 1: Basics of $\mathcal{PT}$-Symmetry
2. Section 2: Linear Algebra
3. Section 3: RF Applications

Preliminaries

Definition of Terms [1], [2]:

Symmetry

A transformation that does not change the value of the Lagrangian (i.e. that does not change the equations of motion such as Maxwell's equations.)

Parity

- Parity Operator $\mathcal{P}$: Mirror inversion $\mathbf{r} \mapsto -\mathbf{r}$
  Cartesian $(x, y, z) \mapsto (-x, -y, -z)$, Spherical $(r, \theta, \phi) \mapsto (r, \pi - \theta, \pi + \phi)$.
  Scalar functions $\psi(\mathbf{r}; t) \mapsto \psi(-\mathbf{r}; t) =: \psi_p(\mathbf{r} : t)$. If ψ is an even (odd) function of $\mathbf{r}$, then $\mathcal{P}\psi_{\substack{e\\o}}(\mathbf{r}; t) = \pm\psi_{\substack{e\\o}}(\mathbf{r}; t)$.

Parity...

Polar vectors change sign $\mathscr{E}(\mathbf{r};t) \mapsto -\mathscr{E}(-\mathbf{r};t) =: -\mathscr{E}_p(\mathbf{r};t)$,
$\mathbf{k} \mapsto -\mathbf{k}$ $\mathscr{D} \mapsto -\mathscr{D}_p$, $\mathscr{J} \mapsto -\mathscr{J}_p$, $\nabla \mapsto -\nabla$, $\boldsymbol{\nabla}\times \mapsto -\boldsymbol{\nabla}\times$.
Axial vectors do NOT change signs $\mathscr{H}(\mathbf{r};t) \mapsto \mathscr{H}(-\mathbf{r};t)$, $\mathscr{B} \mapsto \mathscr{B}$,
$\mathscr{M} \mapsto \mathscr{M}$, $\boldsymbol{\nabla}\times\mathscr{E}(\mathbf{r};t) \mapsto \boldsymbol{\nabla}\times\mathscr{E}(-\mathbf{r};t)$.
Note: Subscript p in fields above means they are space reversed.

Parity...

Eigenvalues of the Parity Operator: $\mathcal{P}\psi = \lambda_P \psi$

$\mathcal{P}\psi(\mathbf{r}; t) = \psi(-\mathbf{r}; t) \implies \mathcal{P}^2\psi(\mathbf{r}; t) = \psi(\mathbf{r}; t) = \lambda_P^2 \psi(\mathbf{r}; t) \implies$
$\lambda_P = \pm 1.$ The corresponding eigenfunctions are either unchanged or change in sign when acted upon by this operator $\mathcal{P}\psi = \pm\psi$.

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If a function is the product two even parts or two odd parts (example $\psi = \cos(k_x x)\cos(k_y y)$ or $\psi = \sin(k_x x)\sin(k_y y)$ the overall function has even parity. On the other hand, if the parts have opposite parity (example $\psi = \sin(k_x x)\cos(k_y y)$), then the overall function has odd parity.

Time Reversal

- Time Reversal Operator $\mathcal{T}$ [3, p. 249]: Run time backwards $t \mapsto -t$. Real-valued scalar field $\psi(\mathbf{r}; t) \mapsto \psi(\mathbf{r}; -t) =: \psi_r(\mathbf{r}; t)$. Real-valued vector fields transform as
$\mathscr{E}(\mathbf{r}; t) \mapsto \mathscr{E}(\mathbf{r}; -t) =: \mathscr{E}_r(\mathbf{r}; t)$, $\mathscr{D}(\mathbf{r}; t) \mapsto \mathscr{D}_r(\mathbf{r}; t)$,
$\mathscr{M}(\mathbf{r}; t) \mapsto \mathscr{M}_r(\mathbf{r}; t)$, $\mathscr{J}(\mathbf{r}; t) \mapsto -\mathscr{J}_r$, $\mathscr{H}(\mathbf{r}; t) \mapsto -\mathscr{H}_r$,
$\mathscr{B}(\mathbf{r}; t) \mapsto -\mathscr{B}_r$, $\mathscr{E}(\mathbf{r}; t) \times \mathscr{H}(\mathbf{r}; t) \mapsto -\mathscr{E}_r \times \mathscr{H}_r$.
$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J} \mapsto \nabla \times \mathscr{H}_r = \frac{\partial \mathscr{D}_r}{\partial t} + \mathscr{J}_r$
$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} - \mathscr{M} \mapsto \nabla \times \mathscr{E}_r = -\frac{\partial \mathscr{B}_r}{\partial t} - \mathscr{M}_r$.
Note: Subscript r in fields above means they are time reversed fields.

Time Reversal Frequency Domain (ω real)

$$\mathscr{E}(\mathbf{r};t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \mathbf{E}(\mathbf{r};\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \mathbf{E}^*(\mathbf{r};-\omega)e^{j\omega t}\,d\omega$$

$$\mathscr{E}_r(\mathbf{r};t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \mathbf{E}(\mathbf{r};\omega)e^{-j\omega t}\,d\omega = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \mathbf{E}(\mathbf{r};-\omega)e^{j\omega t}\,d\omega$$

$\mathscr{E}(\mathbf{r};t)\mapsto\mathscr{E}_r(\mathbf{r};t) \implies \mathbf{E}(\mathbf{r};\omega)\mapsto\mathbf{E}^*(\mathbf{r};\omega)$, ω real, and

$\mathscr{H}(\mathbf{r};t)\mapsto-\mathscr{H}_r(\mathbf{r};t) \implies \mathbf{H}(\mathbf{r};\omega)-\mapsto\mathbf{H}^*(\mathbf{r};\omega)$, ω real.

$\mathcal{T}\left[\cos(\omega t - kz)\right] = \cos(\omega t + kz) \implies \mathcal{T}\left[Ae^{-jkz}\right] = \left[Ae^{-jkz}\right]^* = A^*e^{jkz}$ (Outgoing wave of amplitude $A \mapsto$ incoming wave of amplitude A^*)

Time Reversal of Circuit Quantities

Charge: $q(t) \mapsto q(-t)$

Voltage: $v(t) \mapsto v(-t)$

Current: $i(t) = \frac{dq(t)}{dt} \mapsto -i(-t)$

Resistance: $R \mapsto -R$; $v_R(t) = i_R(t)R \mapsto v_R(-t) = i_R(-t)R$

Inductance: $L \mapsto L$; $v_L(t) = L\, di_L/dt \mapsto v_L(-t) = L\, di_L(-t)/dt$

Capacitance: $C \mapsto C$; $i_C(t) = C\, dv_c(t) \mapsto i_C(-t) = C\, dv_C(-t)/dt$

Time-Reversal Transformation of Scattering Matrix

$\mathbf{b} = S(\gamma, B_0, \omega)\mathbf{a}$, $\gamma =$ loss parameter ($\gtrless 0$ $\substack{\text{loss} \\ \text{gain}}$), $B_0 =$ bias parameter. Incoming wave amplitudes $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$; Outgoing wave amplitudes $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Scattering matrix properties [4]:

(i) $S(\gamma, B_0, \omega) = S^T(\gamma, -B_0, \omega)$ (reciprocity with bias)

(ii) $S^\dagger(\gamma, B_0, \omega)S(-\gamma, B_0, \omega) = \mathbb{I}$ (microscopic reversibility)

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For a lossless system $\gamma = 0$, so that $S(-\gamma, B_0, \omega) = S(\gamma, B_0, \omega)$. Only in that case do we have the unitary property of the scattering matrix: $S^\dagger S = \mathbb{I}$ and eigenvalues of S of the form $e^{j\theta}$, θ real.

Time-Reversal Transformation of Scattering Matrix...

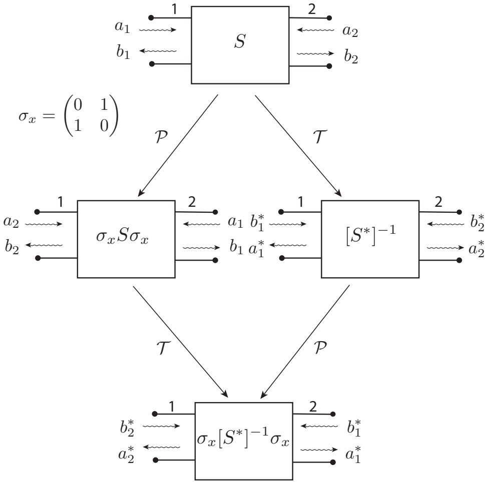
$$\mathcal{T}\mathbf{a} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}^* = \mathbf{b}^*,\ \mathcal{T}\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}^* = \mathbf{a}^* \implies$$

(iii) $\mathcal{T}S(\gamma, B_0, \omega) = [S^*(\gamma, B_0, \omega)]^{-1} = \left[S^\dagger(\gamma, -B_0, \omega)\right]^{-1} = S(-\gamma, -B_0, \omega)$, the latter two using properties (i) and (ii).

Time-Reversal Transformation of Scattering Matrix...

$\mathcal{T}\mathbf{a} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}^* = \mathbf{b}^*, \ \mathcal{T}\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}^* = \mathbf{a}^* \implies$
(iii) $\mathcal{T}S(\gamma, B_0, \omega) = [S^*(\gamma, B_0, \omega)]^{-1} = \left[S^\dagger(\gamma, -B_0, \omega)\right]^{-1} =$
$S(-\gamma, -B_0, \omega)$, the latter two using properties (i) and (ii). If a system is $\mathcal{T}$-symmetric, then $\mathcal{T}S = S \implies$
$S(-\gamma, -B_0, \omega) = S(\gamma, B_0, \omega)$. This is only possible if S is an even function of γ or if $\gamma = 0$ (lossless) and $B_0 = 0$ (no bias). An ordinary passive system comprised of R, L, C is NOT $\mathcal{T}$-symmetric unless $R = 0$. Note $\mathcal{T}^2 = \mathbb{I}$ always.

$\mathcal{PT}$ Transformed Scattering Matrix



Consequences of $\mathcal{PT}$-Symmetry

Note: The parity operation on the scattering matrix results in swapping along both diagonals

$$\mathcal{P}S = S\big|_{\substack{1\to 2\\ 2\to 1}} = \sigma_x S \sigma_x \quad = \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{bmatrix}$$

and $\mathcal{PT}$-symmetry requires

$$\sigma_x \left[S^*\right]^{-1} \sigma_x \quad = \quad \frac{1}{\det S^*} \begin{bmatrix} S_{11}^* & -S_{21}^* \\ -S_{12}^* & S_{22}^* \end{bmatrix} \doteq \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

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Equating, $S_{11}^* = S_{11} \det S^*$, $S_{22}^* = S_{22} \det S^*$, $S_{21} = -S_{12}^* \det S$ and $S_{12} = -S_{21}^* \det S \implies \det S = e^{j\theta}$, $|S_{12}| = |S_{21}|$, $\angle S_{11} = \frac{1}{2}\theta = \angle S_{22}$, $\angle S_{21} + \angle S_{12} = \theta \pm \pi$.

Consequences of $\mathcal{PT}$-Symmetry...

- $\det S = e^{j\theta}\left[|S_{11}||S_{22}| + |S_{12}S_{21}|\right]$, $|\det S| = 1$ implies $|S_{11}S_{22}| = 1 - |S_{12}S_{21}|$
  - If the system has unit transmission $|S_{12}| = 1$ and $|S_{21}| = 1$, then $S_{11} = 0$ or $S_{22} = 0$; the reflection coefficient at one port is zero when the other port is match loaded.

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- For Hermitian systems with the property $SS^\dagger = \mathbb{I}$ (or $[S^*]^{-1} = S^T$), $\mathcal{PT}$-symmetry implies $S = \sigma_x S^T \sigma_x$, which further implies that $S_{11} = S_{22} = 0$ for unit transmission.
  - For non-Hermitian systems, $\mathcal{PT}$ symmetry implies the additional condition $\mathcal{P}\gamma = -\gamma$, that is $\gamma(\mathbf{r}) = -\gamma(-\mathbf{r})$, meaning there is a balance of absorption and amplification in parity related regions.

Outline

Overview of Linear Algebra to understand PT Symmetry

In this linear algebra overview, we will revise:

1. Vector spaces, subspaces, basis
2. Linear independence of vectors, orthogonality
3. Eigen value problems
4. Matrix diagonlization, spectral theorem
5. Similarity transformations
6. Hermitian matrices and properties
7. Non-hermitian and defective matrices
8. Jordan normal form
9. Exceptional points
10. Pseudo Hermiticity and Parity Time symmetry

Vector Spaces

Definition of a vector space (VS) :

1. A set whose elements (called vectors) can be added together and multiplied by scalars.
2. Addition and scalar multiplication must satisfy certain properties (next slide).
3. A vector space is closed under these operations, i.e. after any sequence of these operations the result belongs to the same vector space.
4. Examples: Euclidean space, $\mathcal{R}^n$, complex vector spaces, $\mathcal{C}^n$.

Vector Spaces (contd.)

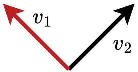
Properties of vector addition and scalar multiplication

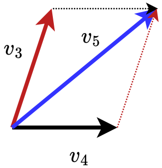
1. Associativity:
   $a + (b + c) = (a + b) + c$
2. Commutativity:
   $a + b = b + a$
3. Identity element: $\exists 0$
   s.t. $a + 0 = a, \forall a \in V$
4. Additive inverse, $\exists - a$
   s.t. $a + (-a) = 0$
5. Identity scalar element:
   $1\,a = a$
6. Distributivity of scalar s:
   $s(a + b) = sa + sb$
7. Distribuitity of scalars s, p:
   $(s + p)a = sp + pa$

Linear independence of vectors

k vectors $\{v_1, \ldots, v_k\}$ are linearly independent iff the following holds: $\boxed{\sum_i c_i v_i = 0}$ only when $\boxed{c_1 = \ldots = c_k = 0}$.

v_1, v_2 linearly independent





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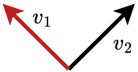

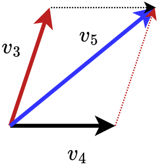

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Stated another way: $$\underbrace{\begin{bmatrix} v_1 & \ldots & v_k \\ \downarrow & \ldots & \downarrow \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ . \\ c_k \end{bmatrix}}_{c} = 0, \text{ iff } c = 0.$$

Basis of a vector space

If a vector space V consists of all linear combinations of $\{v_1, \ldots, v_k\}$, we say that these vectors *span* the space V.

E.g., $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, but also by $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Basis for a vector space

Tightening, say that $\{v_1, \ldots, v_k\}$ form a basis for V if vectors are: linearly independent **and** span V.

Dimension of a vector space: the # of basis vectors

Vector Subspaces

Vector subspace

A subset of a vector space which satisfies the conditions of a vector space (see earlier).

e.g. $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ form (and span) a 2-dim sub-space of $\mathbb{R}^3$.

For $A \in \mathbb{R}^{m\times n}$, think of two types of vectors that it multiplies:

1. $Ax_n = 0$, we say these x_n live in the null space of A: $N(A)$.
2. $Ax_r \neq 0$, we say these x_r live in the row space of A: $R(A)$.

Null and row space of A: examples of vector subspaces of $\mathbb{R}^n$

Orthogonality and Subspaces

Orthogonality between a and b

In the case of $\mathbb{R}^n$: $a^T b = b^T a = 0 = \sum_i a_i\, b_i$

In the case of $\mathbb{C}^n$: $a^H b = (b^H a)^* = 0 = \sum_i a_i^*\, b_i$

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Orthogonality between subspaces

We say that two vector subspaces are orthogonal if *any* pair from the 2 spaces are orthogonal to each other.

[Without proof]: $N(A)$ and $R(A)$ are orthogonal subspaces.

Quiz: if $Ax = 0$ only for $x = 0$, how many vectors are in $N(A)$?

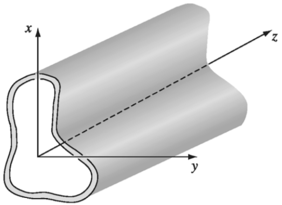
Eigen values and Eigen vectors

The eigen value problem (EVP):

Given a square matrix A, find vectors/scalars x/λ s.t. $\boxed{Ax = \lambda x}$

These occur all the time in electromagnetics, e.g. the waveguide equation: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\frac{\omega}{c})^2 - k^2\right] E_z = 0$.

EVP in disguise: $[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}] \to A$, $E_z \to x$, and $[k^2 - (\frac{\omega}{c})^2] \to \lambda$.



In rectangular/circular geometries, easy to solve directly. However, for arbitrary cross-sections solve EVP.

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3. $\implies A'$ not invertible, i.e. $\det(A') = \det(A - \lambda \mathbb{I}) = 0$.

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E.g. $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ and $\det(A - \lambda \mathbb{I}) = (4 - \lambda)(-3 - \lambda) + 10 = 0$.
Must have 2 (possibly complex) roots, roots are eigenvalues.

Called the *characteristic polynomial* of A.

IMP: A real matrix can have complex eigenvalues.

Algebraic & geometric multiplicities

Algebraic multiplicity

The number of times an eigen value repeats, denoted as $\mu_A(\lambda_i)$

e.g. if $p_A(\lambda) = (\lambda - 1)^2(\lambda - j) = 0$, then $\mu_A(1) = 2$, $\mu_A(j) = 1$.

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Geometric multiplicity

No. of linearly independent eigenvecs for a *given* eigenvalue, $\gamma_A(\lambda_i)$

Clearly, $\boxed{\sum_i \mu_A(\lambda_i) = n}$, $\boxed{1 \leq \sum_i \gamma_A(\lambda_i) \leq n}$.

Also $\boxed{\gamma_A(\lambda_i) \leq \mu_A(\lambda_i)}$ (stated w/o proof).

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E.g. consider identity matrix $\mathbb{I}$. Char. poly. $p(\lambda) = (1 - \lambda)^n = 0$.

Eigs? Eigvecs? $\mu_{\mathbb{I}}(1) = n$, and $\gamma_{\mathbb{I}}(1) = n$

Relation between eigenvectors

Say that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$.

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Such eigenvectors are linearly independent!

For independence, $c_1 v_1 + c_2 v_2 = 0$ only when $c_1 = c_2 = 0$.

(Left multiply A): $c_1\lambda_1 v_1 + c_2\lambda_2 v_2 = 0$

(Multiply 1st eqn by λ_1 and subtr): $c_2(\lambda_1 - \lambda_2)v_2 = 0$

$\implies c_2 = 0$ which leads to $c_1 = 0$, $\implies v_1, v_2$ lin indepn.

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Eigenspace of an eigenvalue of A

$E_\lambda = \{x : (A - \lambda \mathbb{I})x = 0\}$, i.e. subspace spanned by eig vecs $\subseteq \mathbb{C}^n$

Diagonalization of a matrix

Say $A \in \mathbb{C}^{n \times n}$; *assume* has n lin. indepn. eigen vectors $\{x_i\}$:
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$$AS = \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$\implies A = S\Lambda S^{-1} \text{ or } S^{-1}AS = \Lambda$$

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Say $A \in \mathbb{C}^{n \times n}$; *assume* has n lin. indepn. eigen vectors $\{x_i\}$:

Arrange $S = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$, then

$$AS = \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\Lambda}$$

$\implies A = S \Lambda S^{-1}$ or $S^{-1} A S = \Lambda$

Called the diagonalization of A

Note: If eigenvalues distinct, n lin. indepn eigvecs, S^{-1} exists. If eigenvalues repeated, S^{-1} exists **only** if $\gamma_A(\lambda_i) = \mu_A(\lambda_i)$ for all i.

Similar matrices (a.k.a. Similarity transformation)

When A can be diagonalized, we say $A = S\Lambda S^{-1}$. Related:

Two matrices A, B are called **similar** if this holds:
$A = M\,B\,M^{-1}$, assuming M^{-1} exists.

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$A = M\,B\,M^{-1}$, assuming M^{-1} exists.

- A, B share the same eigenvalues:
  Say that $Ax = \lambda x$, then $M\,B\,M^{-1}x = \lambda x$,
  i.e. $B\,\underbrace{M^{-1}x} = \lambda\,\underbrace{M^{-1}x}$.
  $\implies \lambda$ is also an eigenvalue of B with eigenvector $M^{-1}x$!
- A change of basis for a linear transformation shows up as a similarity transformation.

Hermitian Matrices

A is Hermitian if $A^H = A$. Also has special properties:

1. *$x^H Ax$ is real for any x.* How? $(MN)^H = N^H M^H$, so $(x^H \underbrace{Ax})^H = x^H A^H x = x^H Ax \implies x^H Ax \in \mathbb{R}$.

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3. Eig vecs of distinct eig values are orthogonal.
4. For repeated eigen values, $\mu_\lambda(A) = \gamma_\lambda(A)$, i.e. geometric and algebraic multiplicities are always equal.

Spectral theorem

When A is Hermitian, A can always be expressed as $A = V\Sigma V^H$

- $V = \left[v_1 \dots v_n\right]$, cols are eig vectors, need not be unique.
- V: orthogonal matrix, i.e. $V^H V = \mathbb{I}$, $V^{-1} = V^H$ (Unitary)

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  Set of eig values is called the *spectrum* of A.

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Normal matrices

Theorem also applies to matrices that satisfy $A^H A = AA^H$

Only difference: eig values need not be real. Normal matrices are *more general* than Hermitian

Powers of a Hermitian matrix

When A is Hermitian, $A = V\Sigma V^H$, [V unitary, Σ diag, real]

1. Makes it easy to compute A^2. How?

   $A^2 = A\,A = V\Sigma V^H V\Sigma V^H = V\Sigma^2 V^H$.

   And so on, leading to: $A^n = V\Sigma^n V^H$. Note: $(\Sigma^n)_{kk} = \lambda_k^n$.

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   And so on, leading to: $A^n = V\Sigma^n V^H$. Note: $(\Sigma^n)_{kk} = \lambda_k^n$.
2. What about matrix exponential e^{At}? Use Taylor expansion of e^x $(= 1 + x + x^2/2 + \ldots)$ and apply to e^{At}!
   $e^{At} = \mathbb{I} + (At) + \frac{1}{2}(At)^2 + \ldots$.
   Simplifying: $e^{At} = V \begin{bmatrix} e^{\lambda_1 t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n t} \end{bmatrix} V^H$

Non-Hermitian Matrices

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- Eigen vectors may not be orthogonal
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Implications? Recall waveguide equation:

$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\frac{\omega}{c})^2 - k^2\right] E_z = 0$, EVP in disguise:

$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}] \to A$, $E_z \to x$, and $[k^2 - (\frac{\omega}{c})^2] \to \lambda$. If $A^H \neq A$:

$\implies k = \sqrt{\lambda + (\frac{\omega}{c})^2}$, and $\lambda \in \mathbb{C} \implies k \in \mathbb{C}, \implies$ gain/loss!

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Learn via an example: consider $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Eigvals : $\{3, 2, 2\}$

$\rightarrow$ $\lambda_1 = 3$, eigvec: $v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, but,

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$\rightarrow$ $\lambda_1 = 3$, eigvec: $v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, but,

$\rightarrow$ $\lambda_2 = 2$, $\mu_A(2) = 2$, $\gamma_A(2) = 1$ and we can find only one eigvec: $v_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$.

$\rightarrow$ Stuck, as it seems that the matrix can not be diagonlized in the usual way.

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New concept: **Generalized eigenvector** satisfy $(A - \lambda \mathbf{I})^k v = 0$.

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How does this help? Start by asking if there is $(A-\lambda\mathbf{I})^2 v_3 = 0$?

Note: $(A-\lambda\mathbf{I})^2 v_3 = (A-\lambda\mathbf{I})\underbrace{(A-\lambda\mathbf{I})v_3}_{v_2} = 0.$

So we solve for: $(A-\lambda\mathbf{I})v_3 = v_2$ and get $v_3 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$

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→ with these three vectors, form $V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

→ when we apply a "EVD-like" transformation on A, we see that

$J = V^{-1} A V = \begin{bmatrix} \mathbf{3} & 0 & 0 \\ 0 & \mathbf{2} & 1 \\ 0 & 0 & \mathbf{2} \end{bmatrix}$, called the Jordan normal form.

How does the Jordan normal form help?

We saw that $J = V^{-1}AV \Leftrightarrow A = VJV^{-1}$. How to compute e^{At}?

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As it turns out, for repeated eig values, the sub-blocks have size = algebraic multiplicity: $J_2 = \lambda_2 \mathbb{I} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda_2 \mathbb{I}_2 + N_2$.

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N_2 is always nilpotent: $N_2^2 = 0$

$\rightarrow$ Finally we will get $e^{At} = Ve^{Jt}V^{-1} = V \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & \mathbf{te^{2t}} \\ 0 & 0 & e^{2t} \end{bmatrix} V^{-1}$.

Take home message

Hermitian v/s Defective matrices

For Hermitian A, e^{At} will always give pure exponentials:

$e^{At} = \sum_i e^{\lambda_i t} P_i$.

Whereas for defective matrices,

$e^{At} = \sum_i \{\text{polynomials in } t\} \times e^{\lambda_i t} P_i$

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Why does it matter?

For a system of form: $\frac{d\Psi}{dt} = A\Psi$, solution has form $\Psi = e^{At}$.

$\implies$ Hermitian A: exponential behaviour

defective A: non-exponential behaviour.

Exceptional Points (EPs)

Definition

Exceptional points are degeneracies where:

- **Spectral degeneracy**: Two or more eigenvalues coalesce: $\lambda_1 = \lambda_2$
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Impact: as seen, if a matrix goes Hermitian $\rightarrow$ defective, entire time dynamics can change.

Exceptional Points (EPs) – example

Consider $A(\gamma) = \begin{bmatrix} \epsilon + \gamma & \rho e^{j\alpha} \\ -\rho e^{-j\alpha} & \epsilon - \gamma \end{bmatrix}$, with γ variable

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Possibility of EPs? At two points: $\gamma = \pm\rho$

Jordon normal form: $A_{ep} = V \begin{bmatrix} \lambda_{ep} & 1 \\ 0 & \lambda_{ep} \end{bmatrix} V^{-1}$.

$\implies$ as $\gamma \to \rho$, time evolution changes from $e^{\lambda t}$ to $(1 + t)e^{\lambda t}$.

Exceptional Points (EPs) – implications

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How sensitive are eigen values to changes in γ around the EP?

- $\frac{d\lambda_{1,2}}{d\gamma} = \pm \frac{\gamma}{\sqrt{\gamma^2 - \rho^2}}$
- Near the EP, $\gamma \to \rho$, write $\gamma = \rho + \delta$, $|\delta| \ll \rho$.
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Eigenvalues are extremely sensitive to perturbations near an EP

Concept used to enhance sensitivity by operating near an EP.

Connection with Parity-Time Symmetry?

1. Pseudo Hermiticity $\rightarrow$ $A^H = \eta A \eta^{-1}$ for invertible Hermitian η
2. Spectral theorem for Pseudo Hermitian $\rightarrow$ eig vals are either (1) all real, or (2) come in complex conj pairs.
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   - **unbroken** if every eig vector of A also an eigen vector of $\mathcal{PT}$ $\rightarrow$ eig vals are all real
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5. **PT** transition $\rightarrow$ system moves from "all-real" to "conjugate-pairs" regime (e.g. at an EP)

Why $\mathcal{PT}$ symmetry $\implies$ pseudo-Hermiticity?

By defn: $\mathcal{PT}$ symmetric matrix $A \implies (\mathcal{PT})A = A(\mathcal{PT})$

1. Write $(\mathcal{PT}) = PK$: operations $\rightarrow$ parity P, time symmetry K
2. Parity operation satisfies $P^H = P = P^{-1}$

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4. With the above $(PK)A(PK)^{-1} = A \rightarrow$
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   $\rightarrow (P^{-1})^H(A^*)^H P^H = PA^T P = A^H$

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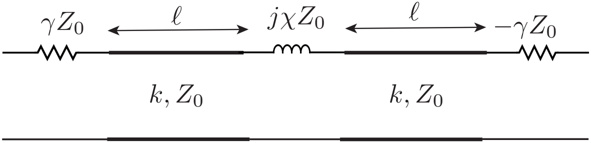
5. A similarity transform is possible in this case: $A^T = SAS^{-1}$
6. Finally $(PA)A(PS)^{-1} = A^H$, pseudo Hermiticity.

Outline

1. Section 1: Basics of $\mathcal{PT}$-Symmetry
2. Section 2: Linear Algebra
3. Section 3: RF Applications

A Balanced Loss-Gain Transmission System



$$\text{Series Impedance}: \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$$

$$\text{Transmission Line}: \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos k\ell & jZ_0 \sin k\ell \\ \frac{j}{Z_0} \sin k\ell & \cos k\ell \end{bmatrix}$$

Figure: A Loss-Gain $\mathcal{PT}$-Symmetric System [5].

Reminder: $ABCD$ matrix relate o/p $\{V, -I\}$ to i/p $\{V, I\}$

Composite Network Parameters

The ABCD-matrix elements of the composite system of Fig 2 are

$$
\begin{aligned}
A &= \cos 2k\ell - j\gamma\chi \sin^2 k\ell + \left(j\gamma - \frac{\chi}{2}\right) \sin 2k\ell = D^* \\
\frac{B}{Z_0} &= j\chi \left(\cos^2 k\ell + \gamma^2 \sin^2 k\ell\right) + j(1-\gamma^2) \sin 2k\ell \\
CZ_0 &= j\left(\sin 2k\ell - \chi \sin^2 k\ell\right)
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\end{aligned}
$$

The corresponding S-matrix elements are

$$
\begin{aligned}
S_{11}(\gamma;\chi;k\ell) &= j\frac{\chi + \gamma(2-\gamma)\left[\sin 2k\ell - \chi\sin^2 k\ell\right]}{(2+j\chi)e^{j2k\ell} - j\gamma^2(\sin 2k\ell - \chi \sin^2 k\ell)} \\
S_{22}(\gamma;\chi;k\ell) &= S_{11}(-\gamma;\chi;k\ell) \\
S_{12} = S_{21} &= \frac{2}{(2+j\chi)e^{j2k\ell} - j\gamma^2(\sin 2k\ell - \chi \sin^2 k\ell)}
\end{aligned}
$$

Eigenvalue Spectrum of Scattering Matrix

- $S_{11}(\gamma;\chi;k\ell)$ is neither an even function nor an odd function of the loss parameter γ and $S_{22} \neq S_{11}$; $S_{11} = 0$ when $\gamma = 1$, $2k\ell = (2p-1), p = 1, 2, \ldots$ and any χ.

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- System is unbiased and reciprocal in the ordinary sense. The transmission parameter $S_{12}(\gamma, \chi, k\ell)$ is an even function of γ.

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Eigenvalues λ of the scattering matrix given by

$$\lambda_{\frac{1}{2}} = \frac{S_{11} + S_{22}}{2} \pm \frac{\sqrt{(S_{11} - S_{22})^2 + 4S_{12}S_{21}}}{2}.$$

The two roots will coincide if $(S_{11} - S_{22}) = \pm 2j\sqrt{S_{12}S_{21}}$.

Eigenvalues of Scattering Matrix

Here the two roots are

$$\lambda_{\frac{1}{2}} = \frac{1}{\text{Den}}\Bigg[j\left(\chi - \gamma^2\left(\sin 2k\ell - \chi\sin^2 k\ell\right)\right)$$

$$\pm\, 2\sqrt{1-\gamma^2\left(\sin 2k\ell - \chi\sin^2 k\ell\right)^2}\Bigg]$$

$$\text{Den} = (2+j\chi)e^{2jk\ell} - j\gamma^2\left(\sin 2k\ell - \chi\sin^2 k\ell\right).$$

Eigenvalues of Scattering Matrix

Here the two roots are

$$\begin{aligned} \lambda_{\frac{1}{2}} &= \frac{1}{\text{Den}}\Bigg[j\left(\chi - \gamma^2\left(\sin 2k\ell - \chi\sin^2 k\ell\right)\right) \\ &\qquad \pm 2\sqrt{1-\gamma^2\left(\sin 2k\ell - \chi\sin^2 k\ell\right)^2}\Bigg] \\ \text{Den} &= (2+j\chi)e^{2jk\ell} - j\gamma^2\left(\sin 2k\ell - \chi\sin^2 k\ell\right). \end{aligned}$$

Exceptional points (EPs) in the spectrum of the Loss-Gain system occur when the two roots coincide, which happens when

$$\gamma(\sin 2k\ell - \chi\sin^2 k\ell) = \pm 1 \implies \gamma = \frac{\pm 2\cos\theta_\chi}{\chi\left[\cos\theta_\chi - \cos(2k\ell - \theta_\chi)\right]},$$

Existence of Exceptional Points

where $\cos\theta_\chi = \chi/\sqrt{\chi^2+4}$.

- No EP if $\gamma = 0$; a balanced combination of gain and loss is needed in the system for the existence of EPs.
- No EP if $\ell = 0$; the phase shift offered by a finite length transmission line is essential for the transition to take place from the unbroken to broken (or vice-versa) regions.

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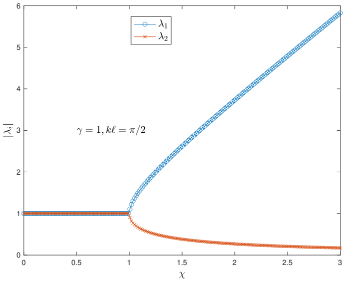
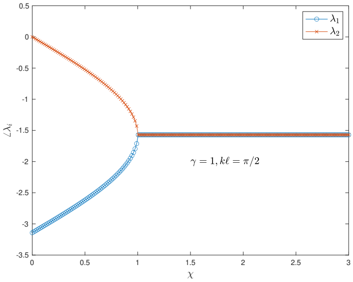
- EP still possible with $\chi = 0$. In this case $2k\ell \neq m\pi$, $m = 1, 2, \ldots$. For e.g., $k\ell = \pi/4$, $\gamma = 1$ defines a valid EP.
- For $k\ell = \pi/2$, EP exists at $\gamma\chi = \pm 1$.

Eigenvalue Spectrum for $\gamma = 1$



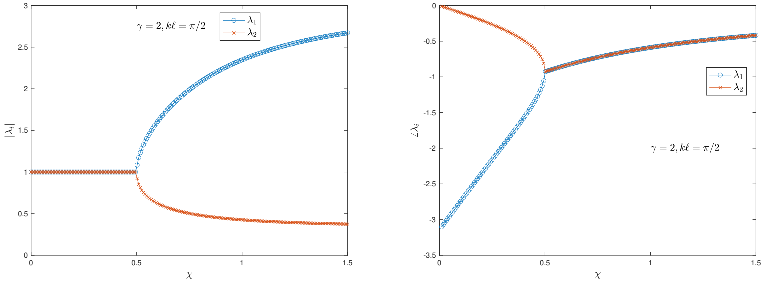
(a) Magnitude $|\lambda_i|$



(b) Phase $\angle\lambda_i$

Figure: Existence of EP at $\chi = \gamma^{-1} = 1$.

Eigenvalue Spectrum for $\gamma = 2$

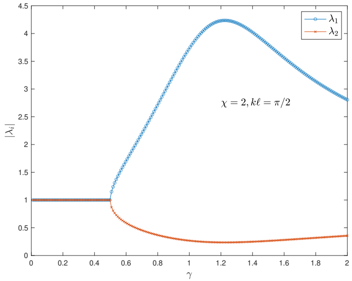


(a) Magnitude $|\lambda_i|$

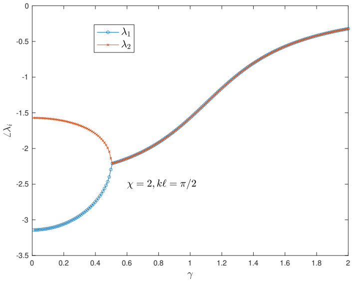
(b) Phase $\angle\lambda_i$

Figure: Existence of EP at $\chi = \gamma^{-1} = 0.5$.

Eigenvalue Spectrum for $\chi = 2$



(a) Magnitude $|\lambda_i|$



(b) Phase $\angle\lambda_i$

Figure: Existence of EP at $\gamma = \chi^{-1} = 0.5$.

Scattering Matrix Elements for $\chi = 2$

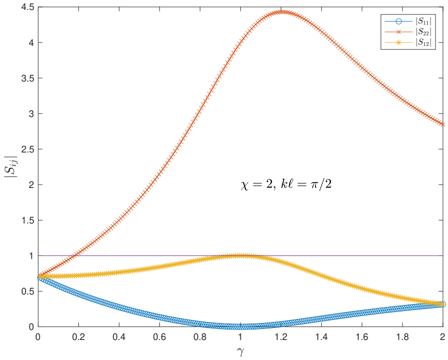


Figure: Matrix element magnitudes $|S_{ij}|$. At the exceptional point $|S_{11}| = 0, |S_{12}| = 1$.

$|S_{11}|$ for $\mathcal{PT}$-Symmetric (S_{11}^{PT}) vs $\mathcal{P}$-Symmetric (S_{11}^{P}) Networks

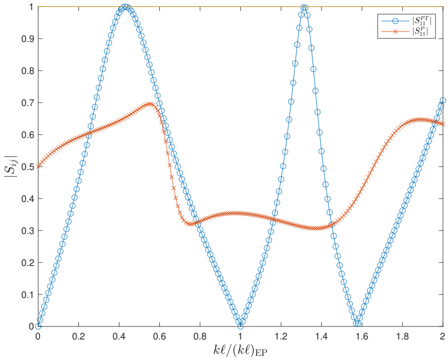


Figure: Absence of exceptional point for $\mathcal{P}$-symmetric only network,

Eigenvalues of Dissipation Matrix

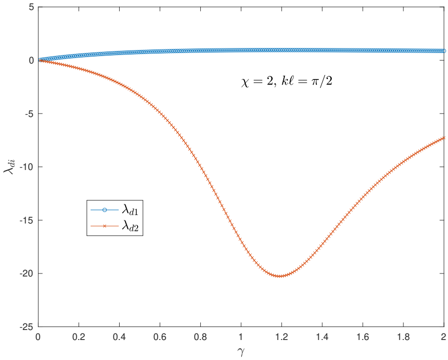


Figure: Eigenvalues λ_{di} of the dissipation matrix $H = \mathbb{I} - SS^\dagger$, $\lambda_{d1} > 0$ (loss), $\lambda_{d2} < 0$ (gain).

Frequency Dependence of S-Matrix Eigenvalue Spectrum

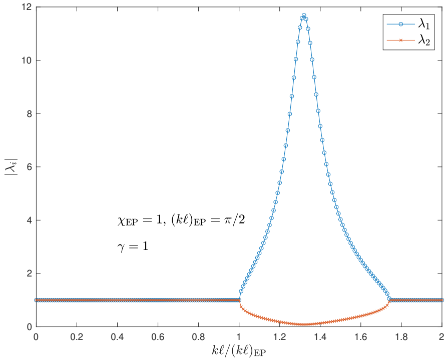


Figure: Eigenvalue spectrum versus normalized frequency. At the center frequency $\chi_{\mathrm{EP}} = 1$, $\gamma = 1$, $(k\ell)_{\mathrm{EP}} = \pi/4$.

Frequency Dependence of Scattering Matrix Elements

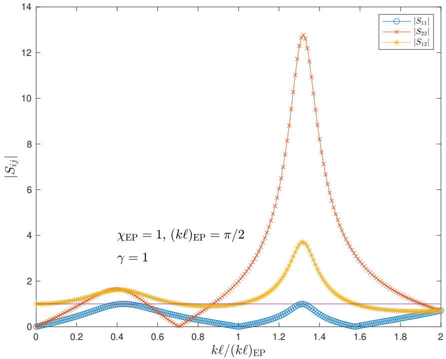


Figure: Matrix element magnitudes $|S_{ij}|$ as a function of normalized frequency. At the center frequency $\chi_{\rm EP} = 1$, $\gamma = 1$, $(k\ell)_{\rm EP} = \pi/4$.

Coupled Loss-Gain Metasurfaces for LWA Design

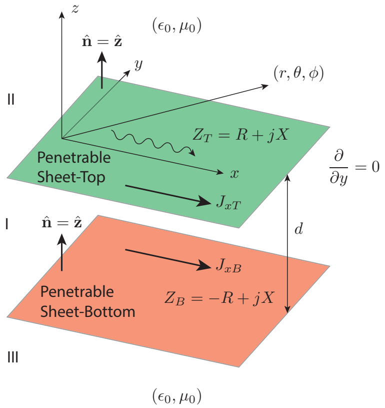
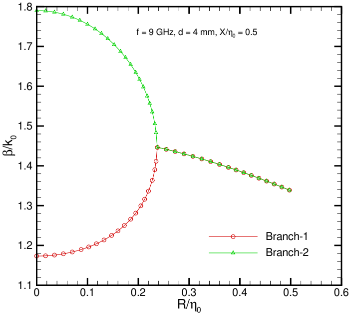
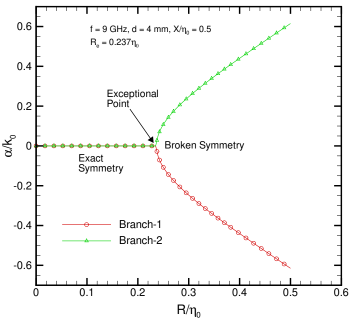


Figure: A Loss-Gain $\mathcal{PT}$-Symmetric System [8], [9].

Propagation Constant for Coupled Metasurfaces



(a) Phase constant β/k_0.



(b) Attenuation constant α/k_0.

Figure: Existence of EP for coupled metasurfaces.

Patterns of Coupled Metasurface LWA at EP

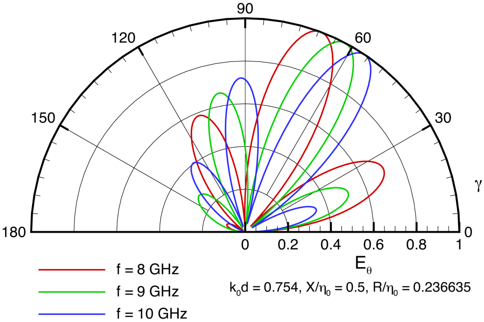


Figure: Radiation patterns of the coupled loss-gain metasurface leaky wave antenna operating at exceptional point [9].

Patterns of Coupled Metasurface LWA at Non-EP

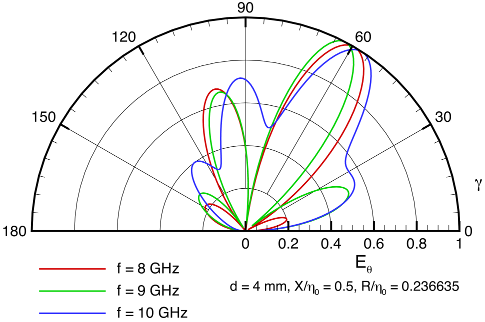


Figure: Radiation patterns of the coupled gain-loss system with exceptional point at 9 GHz and non-exceptional points at 8 GHz and 10 GHz [9].

Radiation Patterns of an Ordinary LWA

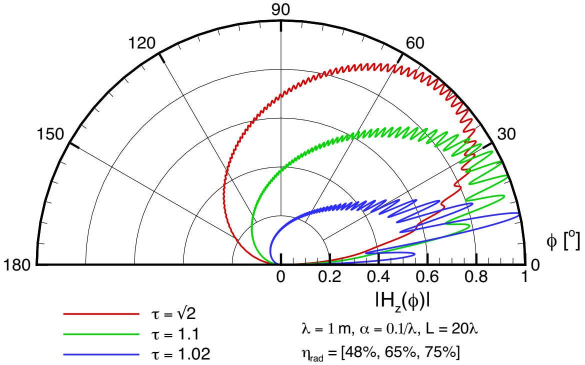


Figure: Radiation patterns of an ordinary LWA comprised of an impenetrable impedance surface [9].

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