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Surface equivalence theorems

1936 → Schelkunoff, building on Huygens's principle.
Based on the uniqueness theorem (Recall: fields in a lossy medium uniquely specified by the sources plus one of the 3 boundary condns: ① E_{tan} over S, ② H_{tan} over S ③ Combo over S.)

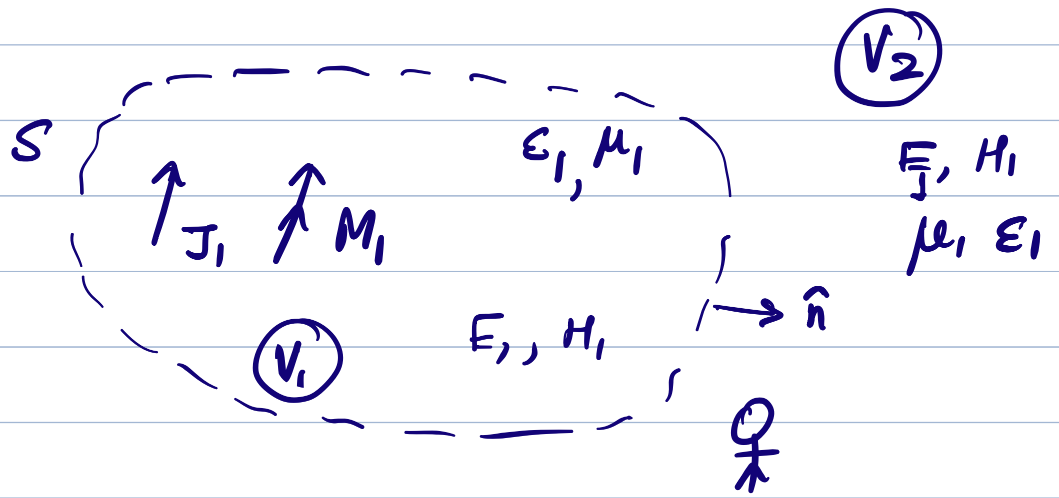
Consider the original problem:

<u>step 0</u>: as-is

$\uparrow J_1 \quad \uparrow M_1$ $\qquad \epsilon_1, \mu_1$

E_1, H_1 everywhere.

<u>step 1</u>: imaginary surface S introduced.



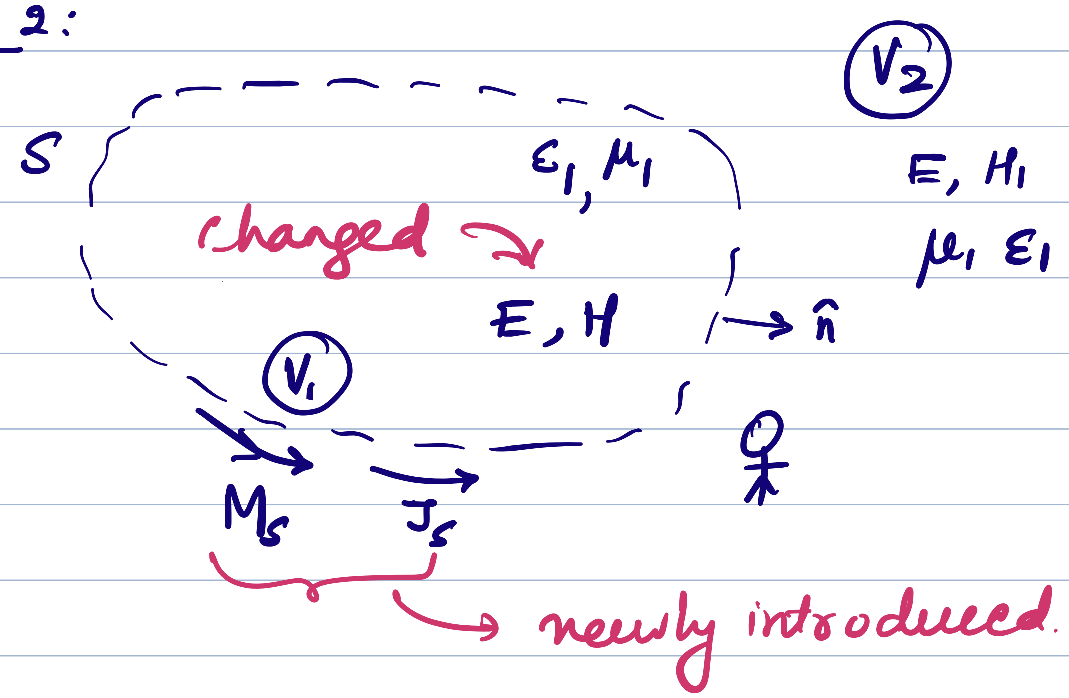
Now ask: In V_2, what BC can observer see?

$\hat{n} \times (H_1 - H_1) = 0$ and $(E_1 - E_1) \times \hat{n} = 0$.

in other words $H_{tan} = n \times H_1$ and $E_{tan} = \vec{E} \times \hat{n}$

Also the obsr has no way to know what caused these fields, ie. J_1, M_1.

Step 2:



a) Again ask: what does obsv see?

$$H_{tan} \left(= \hat{n} \times H_1\right) = \hat{n} \times H + J_s$$
$$E_{tan} \left(= E_1 \times n\right) = E \times \hat{n} + M_s$$

b) what values of M_s & J_s must be chosen such that the obsv can't tell the difference between step 1 & step 2?

obviously if $J_s = -n \times H + n \times H_1$
and $M_s = -E \times n + E_1 \times n$.

Then the tan fields seen by obsv are the same $\Rightarrow$ fields everywhere in V_2 are the same as before via the uniqueness thm.

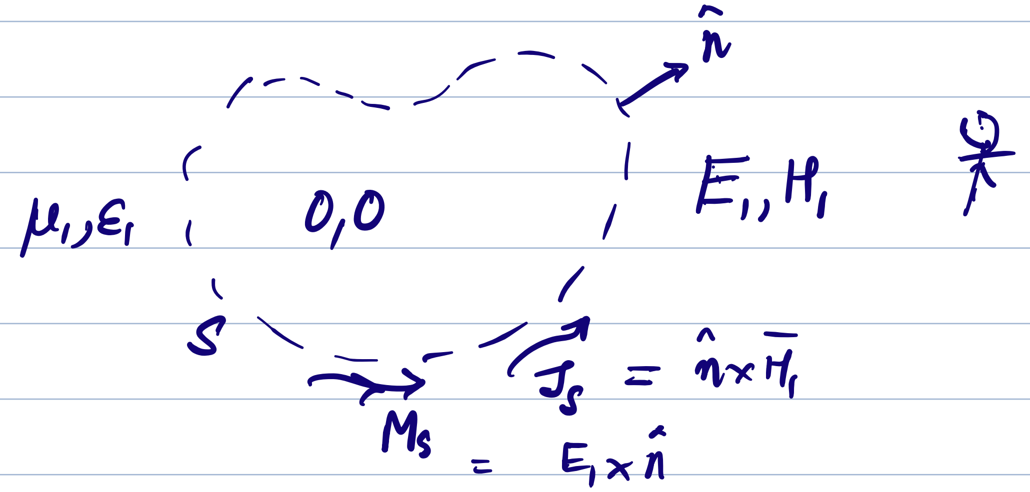
This is called the equivalent problem for V_2 (not V_1) since the fields in V_2 are unchanged.

↳ Let's refine this further. Uniqueness thm says that we need one of E_{tan}, H_{tan}, not both, over S, ie. specifying both entirely over S is not necessary!

Moreover, by changing J_s, M_s, we can alter E, H → ultimately we are interested in V_2 not V_1. We can even alter μ_1, ϵ_1 by the same logic.

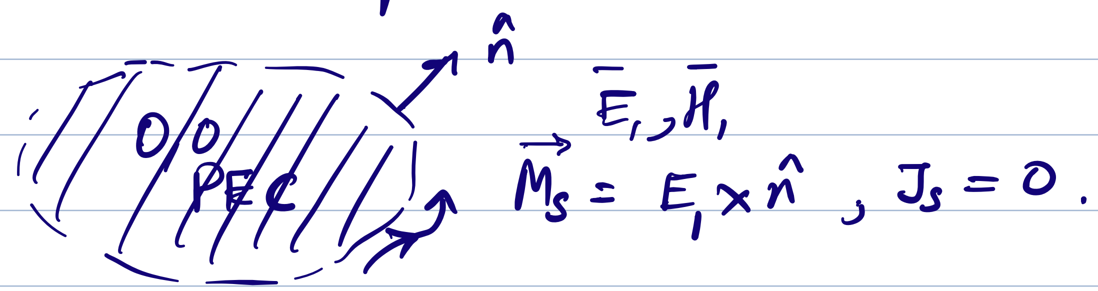
So as a first step consider Love's equivalence principle. Here, we set $E = 0 = H$.

⇒



Here we still have both J_s & M_s. ie. both E_{tan} and H_{tan}, so we can further specialize it.

① Imagine that we place a PEC inside S.

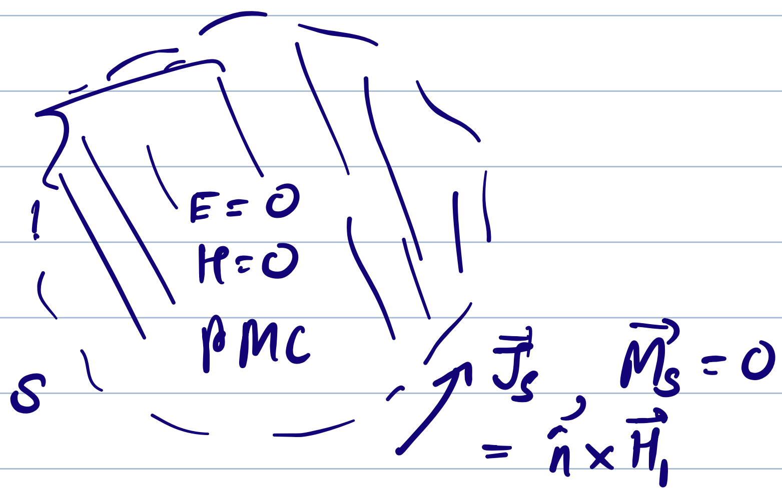


We have already seen that J_s doesn't radiate any field over a PEC, so it's useless to consider it for a radiation problem.

Note: when we have an impressed M_s, the tangential BC on a PEC is <u>NO</u> longer $E_{tan} = 0$. Instead it is $n \times E = -M_s$.
i.e. $E_{tan} = -M_s$

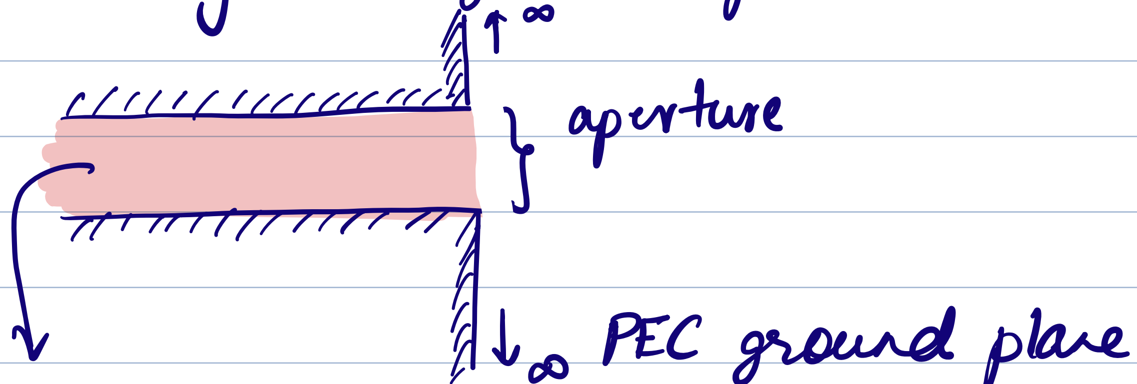
We compute the M_s value from the original problem and impress it on S. Thus we only specified E_{tan} in this problem.

↳ Similarly there is a magnetic equivalent:
Fill the volume S with a PMC.
→ We know that an M_s over a PMC doesn't radiate (can be shown using the reciprocity theorem).



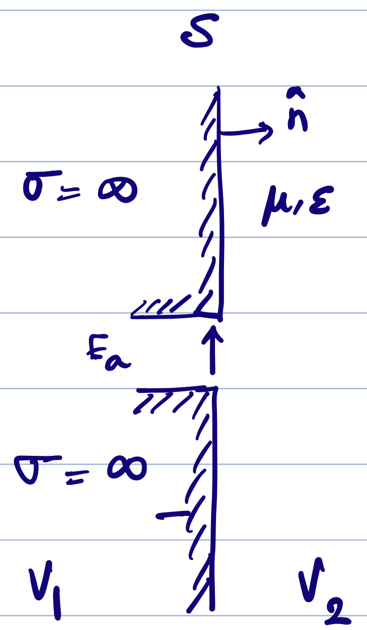
These are equivalent problems → not easier to solve compared to the original problems.

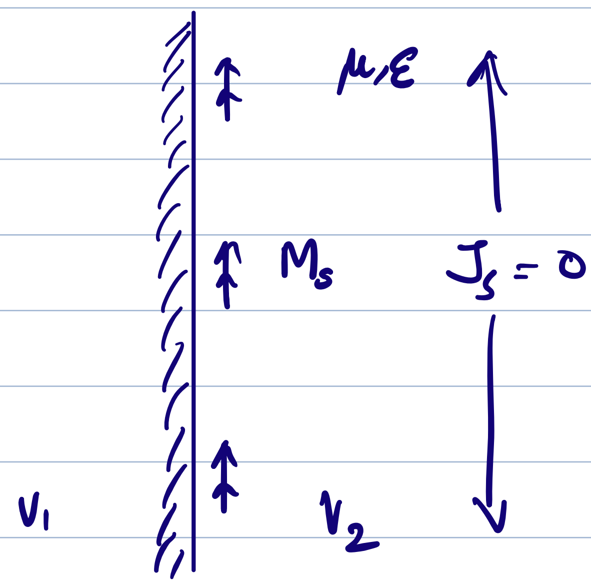
↳ example of how to use these ideas: (book prob)
a waveguide aperture, i.e. a waveguide hitting an ∞ ground plane with an aperture



$\parallel$ plate waveguide.

↳ Say that E_{tan} over aperture is known $= E_a$.



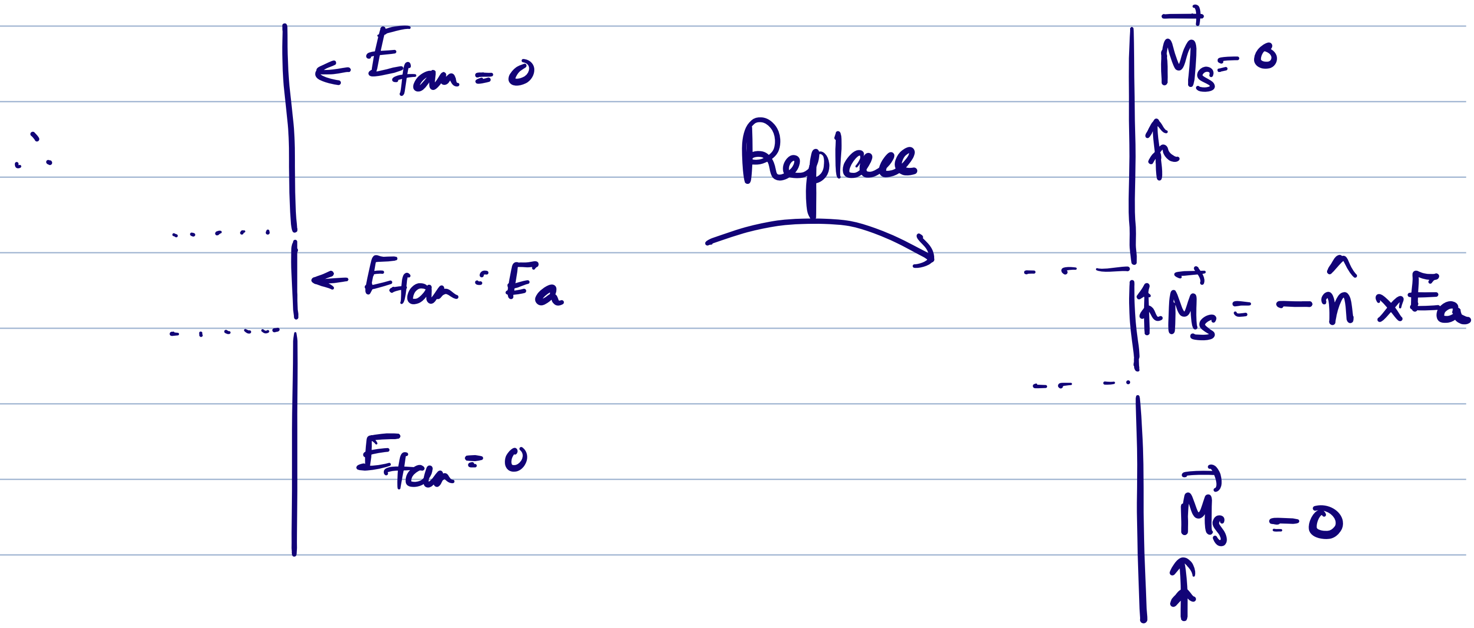


physical problem

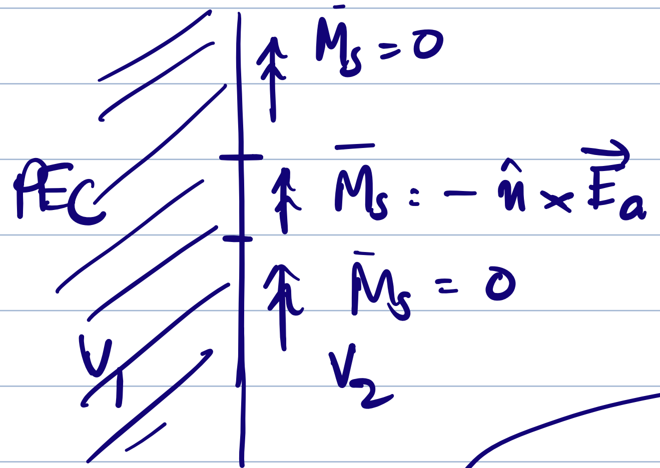
PEC equivalent

want to go to

To do this we need physical values of E_{tan} everywhere.



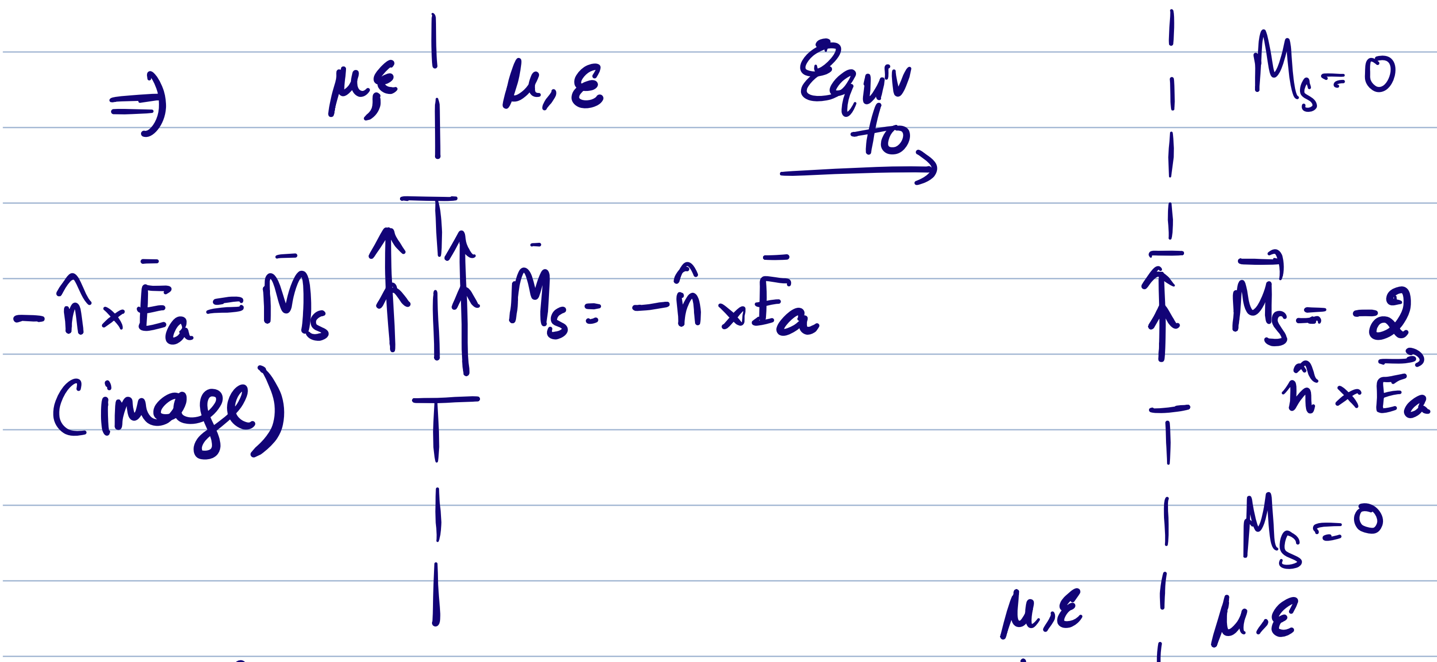
The problem looks like :



Next trick?

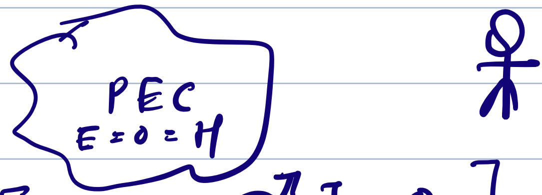
↓

Method of images



Radiating in free space! We know how to solve.

↳ A critical review of the PEC equivalent problem:



<u>Some facts</u>: $\left[\ \vec{M}_s \sim \vec{J}_s = 0\ \right]$ impressed currents.

① Reciprocity thm is derived assuming J/M are impressed sources. Induced currents get included via medium conductivity into a complex ε_r.

② Boundary condition eqns for $\hat{n} \times \Delta H$ or $\Delta E \times \hat{n}$ involve total surface currents, whether they be impressed or induced.

[→ These facts are clear from first principal derivations from Maxwell's equations.]

$\boxed{\text{Case A}}$

So what happens when we put a $J_{surf,imp} \equiv J_{sim}$ on a PEC?

ⓐ It doesn't radiate (from ① above)

ⓑ Can it lead to a $J_{surf,ind} = J_{sid}$?

What does the B.C. say? $\hat{n} \times (H_{V_2} - H_{V_1}) = \vec{J}_{surf}$

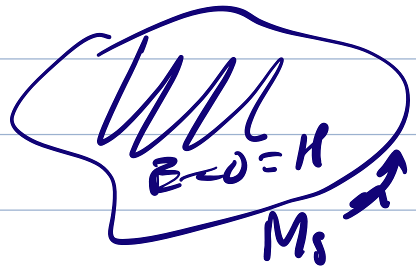
Inside the PEC, $H_{V_1} = 0$, so:

$\hat{n} \times (H_{V_2}) = \vec{J}_{surf} = \vec{J}_{sim} + \vec{J}_{sid}$. Thus $\bar{J}_{sid} = -\bar{J}_{sim}$ to give $H_{2,tan} = 0$, consistent with the fact that a PEC doesn't radiate.

Summary: Jimp → Jind = −Jimp → No radiation.

$\boxed{\text{case b}}$

↳ Now what happens if we have a pure M_s?



on the surface
$E_1 \times \hat{n} = \vec{M}_s$

$\Rightarrow$ there is an H_1 that comes along with E_1.

$\therefore$ The H-BC gives $n \times (H_1 - 0) = \vec{J}_{tot}$

$\Rightarrow$ there must be an induced $\bar{J}$ on the PEC.

i.e. $\vec{J}_{tot} = \vec{J}_{s,ind}$. In fact we have got the value of the induced current: $\vec{J}_{ind} = \hat{n} \times H_1$ where H_1 is the original H field!

Remember that this $\vec{J}_{ind}$ is not a new source, it is dependent on $\vec{M}_s$.
and the fields are determined ultimately by the uniqueness thm by considering M_s.

<u>Summary</u>: $M_s \rightarrow \begin{Bmatrix} E \\ H \end{Bmatrix} \rightarrow J_{ind} = n \times \vec{H}_1$

↳ Finally <u>case c</u> : J_s & M_s.

We don't have to work this out once again. We just use superposition of both cases to get that J_s doesn't radiate, only M_s does.
$\vec{J}_{ind}$ changes.