Image theory

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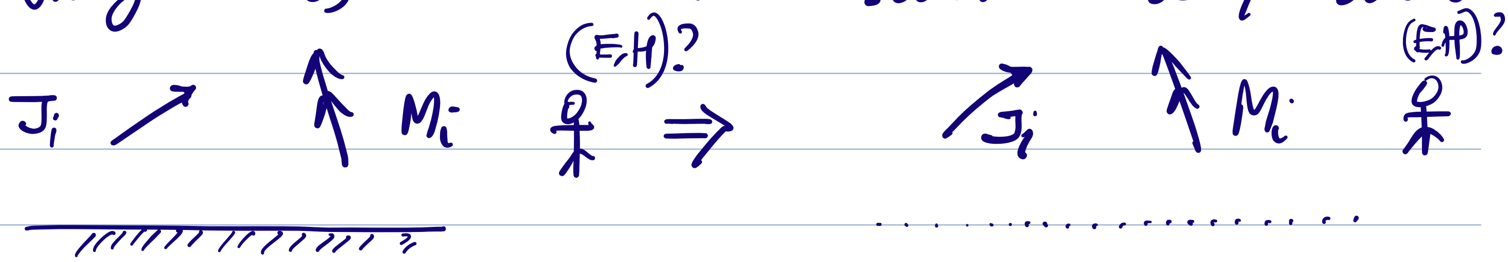
Background: ① we've already learnt this in high school / beginning UG ↳ a charge over a metal (perfect) plane can be represented as a charge and its image.

② We will now upgrade this to current.

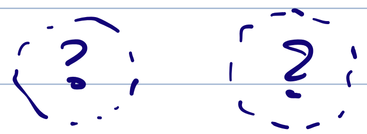
③ Since we are familiar with magnetic currents, we will include them.

In general, we want to answer these questions:

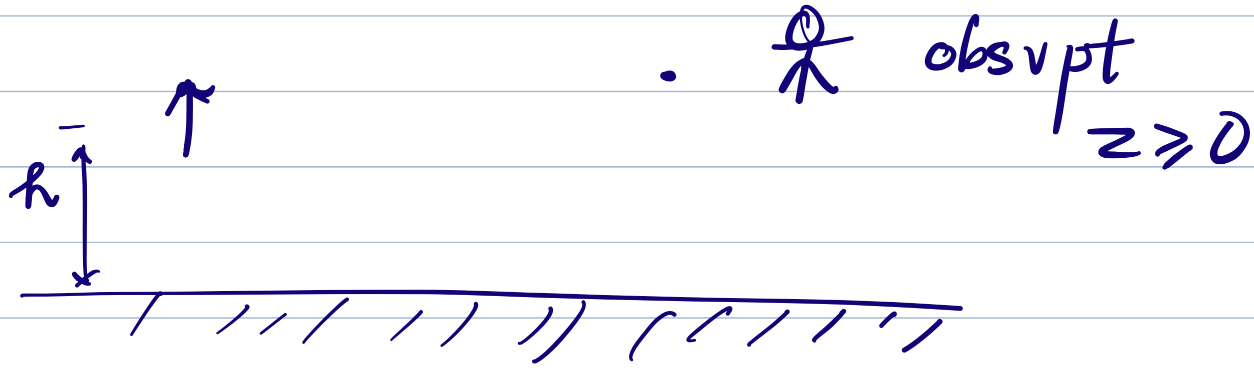


Key pt: Can we get the same fields just above GND?

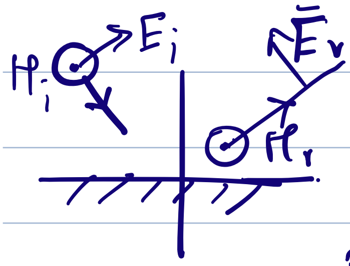
① Consider only far fields
② No mutual coupling



① Start with an electric dipole (vertical) placed over infinite perfect ground

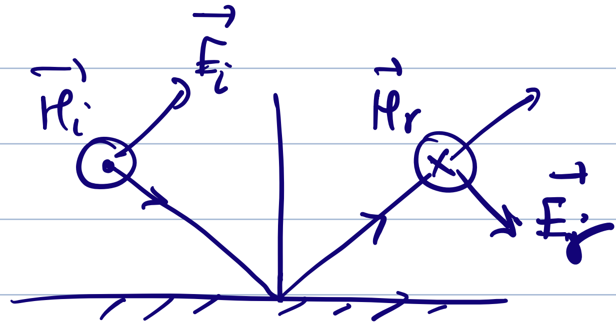


First : how is a plane wave reflected by a PEC?



$\Rightarrow \quad \Gamma = 1$

But be careful:

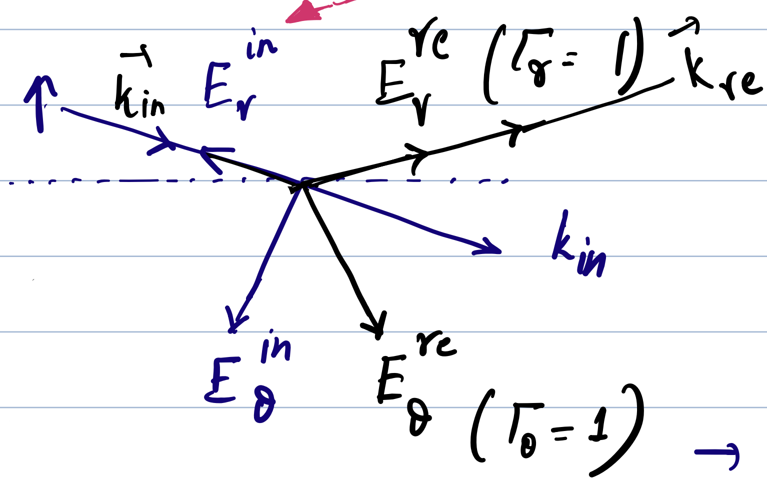


→ Physically these are the same but the sign of Γ depends on how you fix the direction of the reflected fields.

here $\Gamma = -1$.

Recall: $E_r \propto \cos\theta \ \hat{r}$
$E_\theta \propto \sin\theta \ \hat{\theta}$
here $\theta > 90^\circ \Rightarrow E_r < 0$

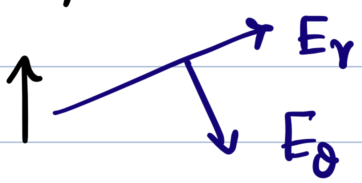
Now, a dipole above ground.



This is the configuration that ensures net tangential $E = 0$

→ Must be done separately for both r, θ components because $|E_r| \neq |E_\theta|$ in general.

So if we want to keep this picture intact:

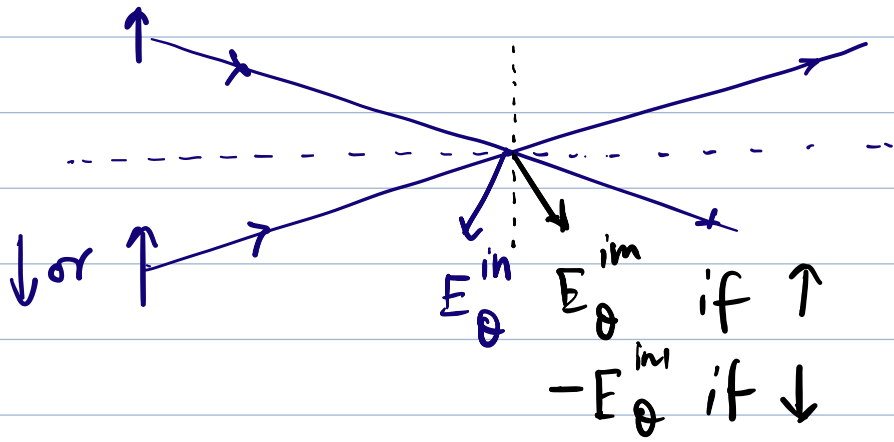


then, $\Gamma = 1$

[Balanis only talks about far field picture, so $E_r = 0$ & the only discussion involve Γ_θ]

< we also focus now on the far field >

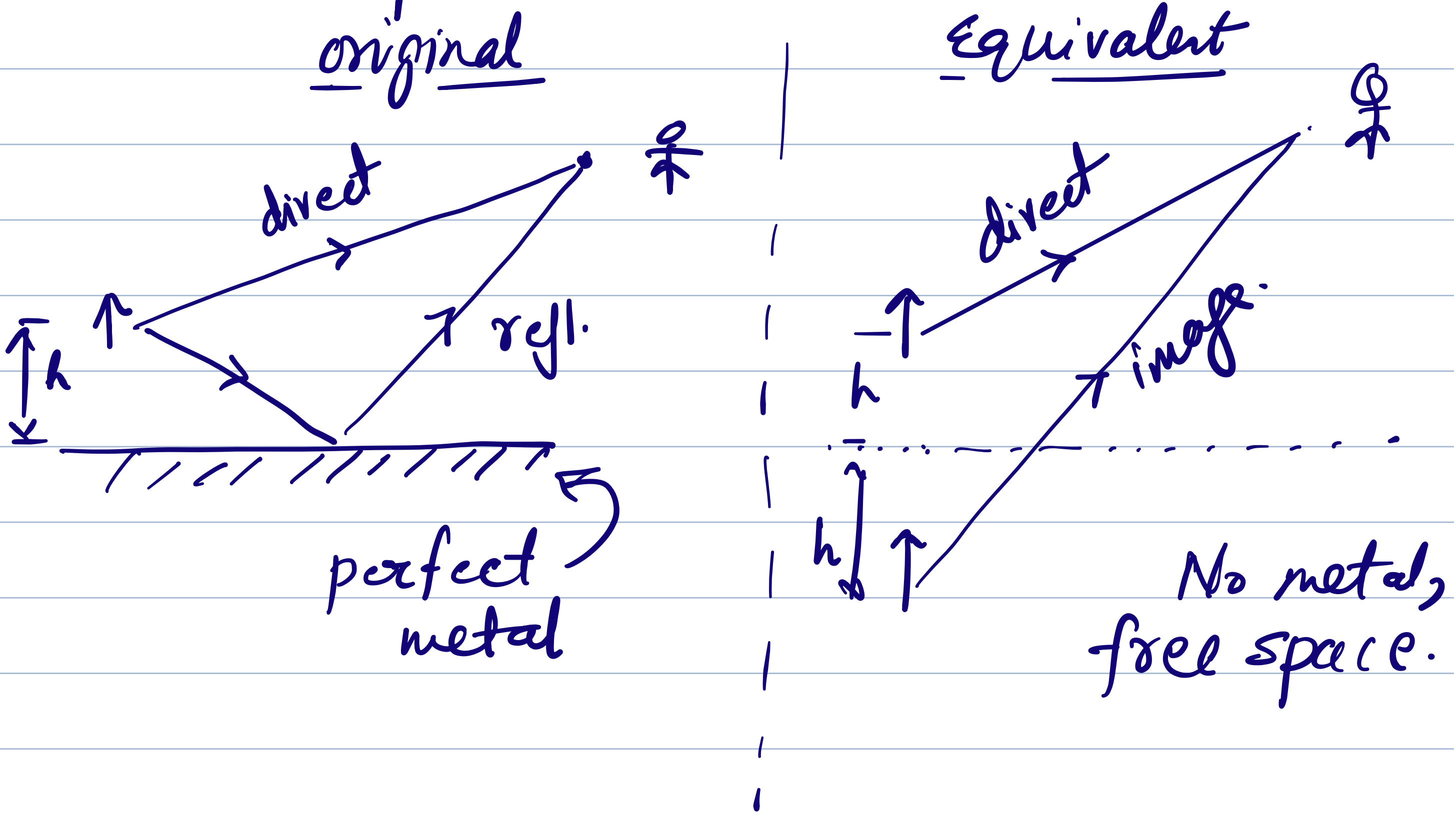
Now suppose we want to replace the metal, we can introduce an image.

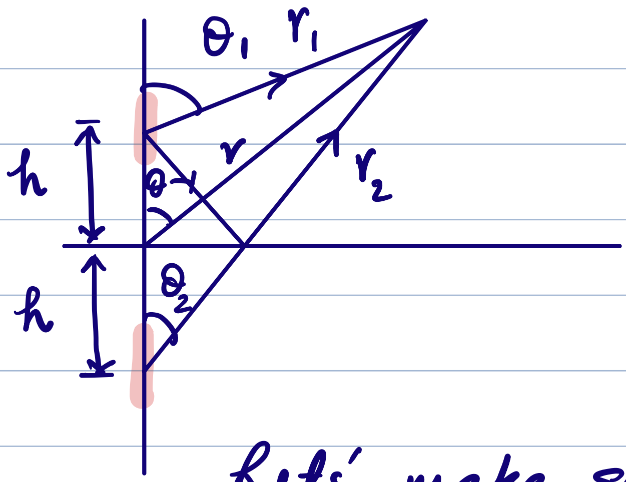


This one is correct as it gives $E_{tan} = 0$

So we can replace the metal with the image and as long as we dive in $\underline{z \geq 0}$, we wouldn't know the difference!

∴ The recipie?





This is the exact geometry, but we want to go to the far field.

Let's make some approximations.

$$r_1^2 = h^2 + r^2 - 2hr\cos\theta = r^2\left[1 + (h/r)^2 - 2h/r\cos\theta\right]$$

$$r_2^2 = h^2 + r^2 + 2hr\cos\theta = r^2\left[1 + (h/r)^2 + 2h/r\cos\theta\right]$$

far field : $h << r$.

$\therefore$ for $|r_1|$ terms : $r_1 \approx r$, $r_2 \approx r$

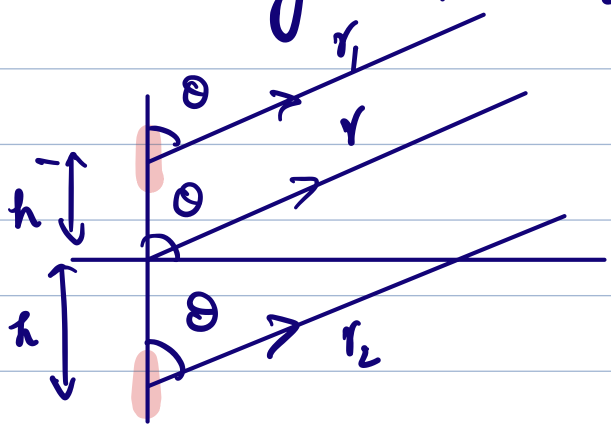
phase terms : $r_1 \approx r\left[1 - \frac{2h}{r}\cos\theta + \left(\frac{h}{r}\right)^2\right]^{1/2}$

$$\approx r\left[1 - \frac{1}{2}\left(\frac{2h\cos\theta}{r}\right)\right]$$ binomial exp

$$= r - h\cos\theta$$

Similarly $r_2 \approx r + h\cos\theta$.

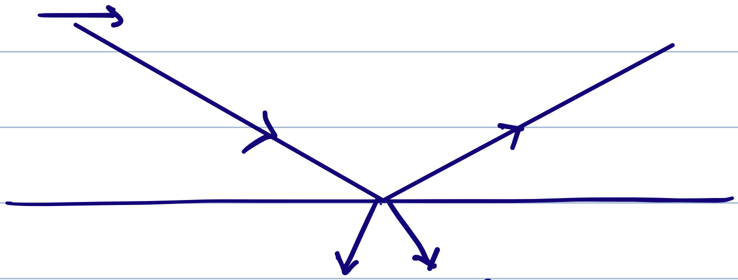
Similarly $\theta_1 \approx \theta_2 \approx \theta$



Now the individual components can be added correctly.

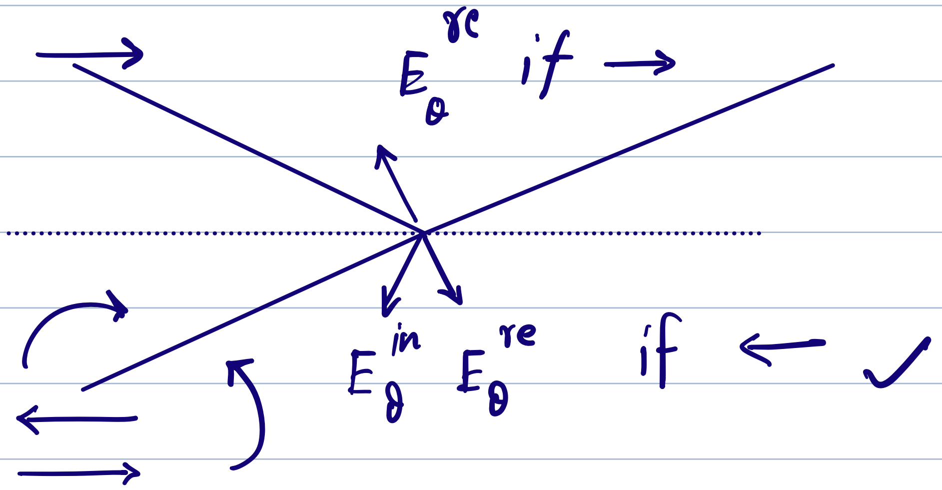
Next is a horizontal dipole. Now we got the hang of it.

original:



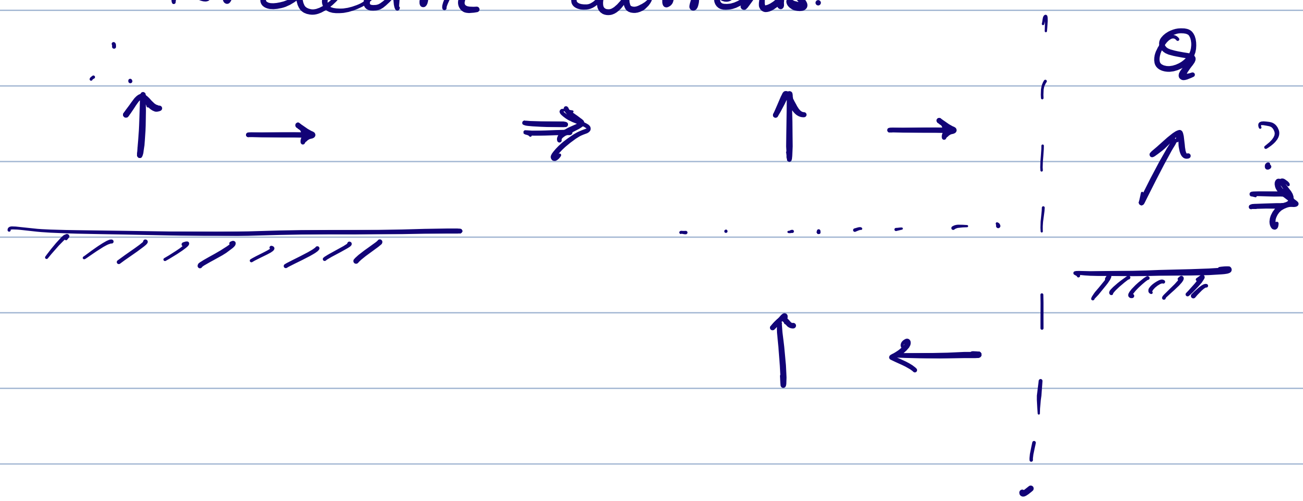
E_θ^{in} E_θ^{re} $\rightarrow$ Net $E_{tan} = 0$

Equivalent:



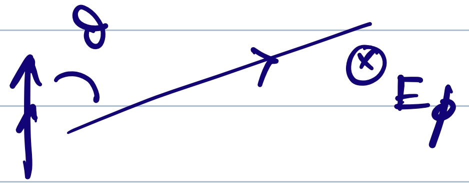
This gives us $E_{tan} = 0$.

For electric currents.

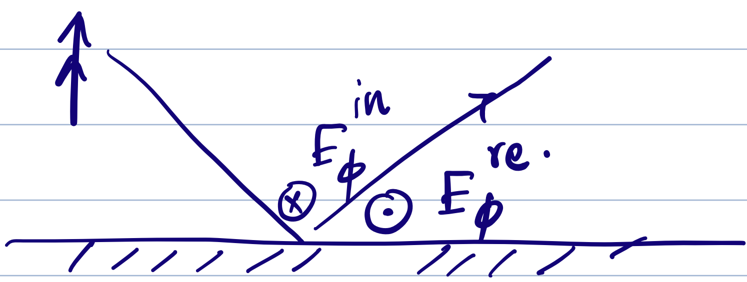


We can easily work out now what happens to magnetic current.

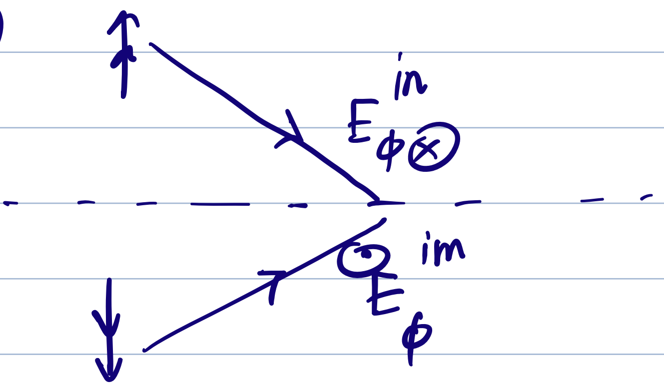
Recall $\Uparrow \vec{M}$ has E_ϕ, H_r, H_θ.



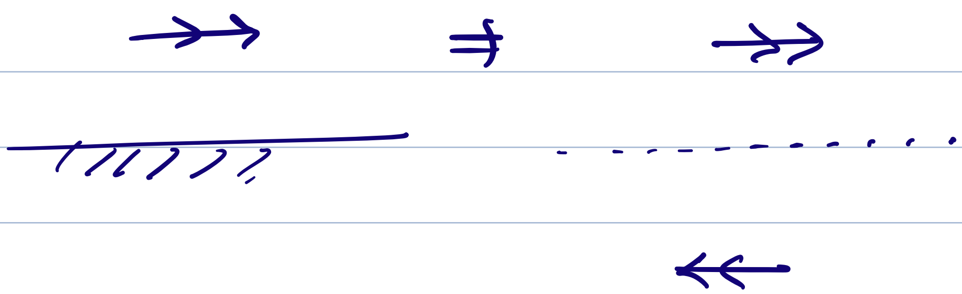
∴ when placed over ground:



$\Rightarrow$

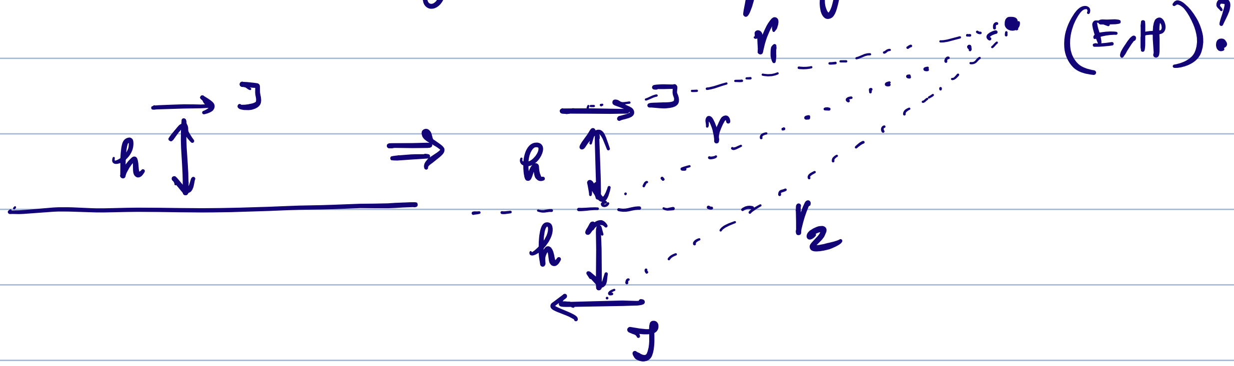


Similarly



HW: Work out images for all 4 cases when the reflector is a PMC (perfect magnetic cond)

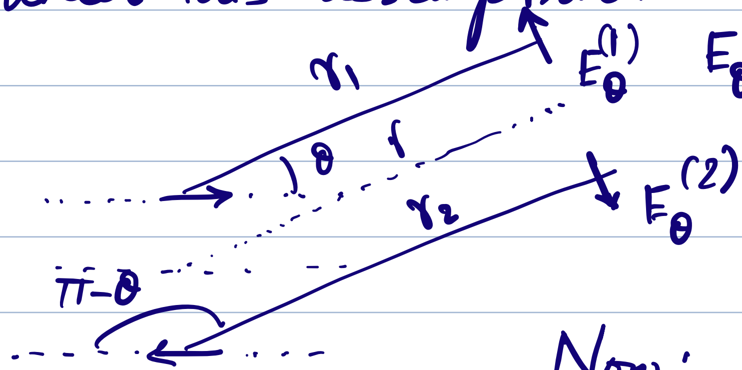
↳ An imp exercise now:
consider a $\vec{J}$ just above perfect metal.



Let's assume $r_1 \approx r_2$ and $h << r$
Under this assumption:

(Hertz dipole)

$$E_\theta = j\eta \frac{k I_0 l \sin\theta}{4\pi r} e^{-jkr}$$



Now:

$$E_\theta^{(1)} = j\eta \frac{k I_0 J}{4\pi r} \sin\theta\, e^{-jk(r - h\cos\theta)}$$

$$E_\theta^{(2)} = -j\eta \frac{k I_0 J}{4\pi r} \sin\theta\, e^{-jk(r + h\cos\theta)}$$

But as $h \to 0$, $E_\theta^{(1)} = -E_\theta^{(2)}$, perfectly cancelling.

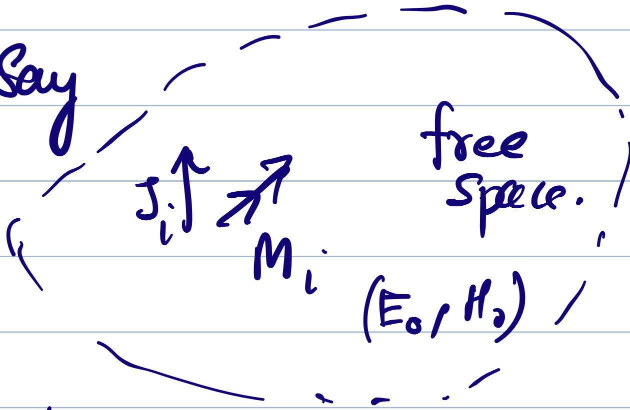
IMP conclusion: placing a horizontal current over a PEC produces no radiation, i.e. useless as an antenna!

(OTOH, a horizontal M_s is good → adds up constructively.)

Volume Equivalence Theorem.

This is an elaboration of Huygen's principle.

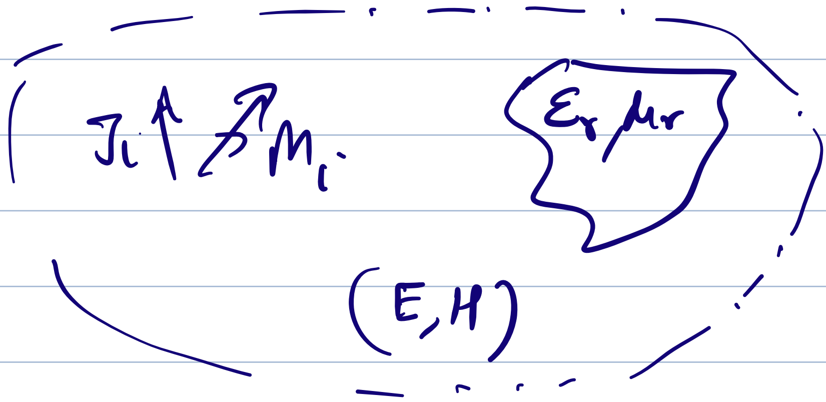
Say



Then, $\nabla \times E_0 = -M_i - j\omega\mu_0 H_0$,

$\nabla \times H_0 = J_i + j\omega\varepsilon_0 E_0$ — ①

Now, the same currents illumine an object



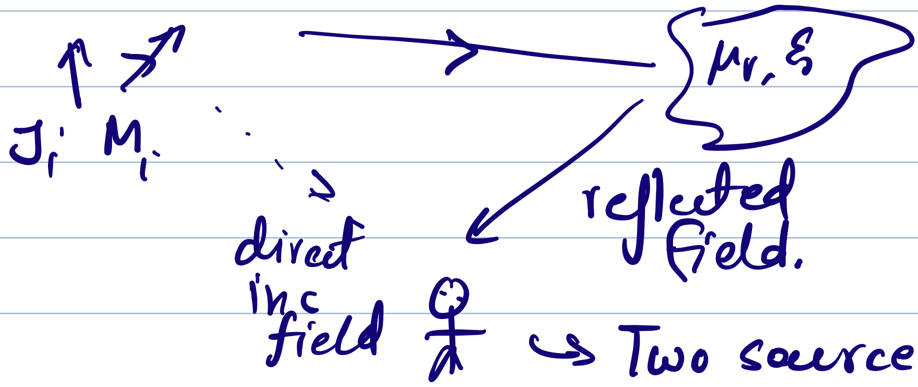
Now:

$\nabla \times E = -M_i - j\omega\mu H$

$\nabla \times H = J_i + j\omega\varepsilon E$ — ②

<u>Question</u>: can I map the above scenario onto a case that looks like free space propagation, maybe with some new currents? [like we did in image theory]

Note what is happening physically



Total = primary + secondary.

↪ Two sources of fields

Let's subtract the two sets: Then,

$$\nabla \times (E - E_0) = -jw[\mu H - \mu_0 H_0] \quad \&$$
$$\nabla \times (H - H_0) = +jw[\varepsilon E - \varepsilon_0 E_0]$$

Introduce $E_s = E - E_0$, $H_s = H - H_0$ [scattered fields]

I want it to look like free space, so I need
① $E - E_0$ on both LHS & RHS and
② ε_0, μ_0 on the RHS.

So $$\nabla \times (E - E_0) = -jw[\mu H \underbrace{- \mu_0 H + \mu_0 H}_{} - \mu_0 H_0]$$

$$\nabla \times (E - E_0) = -jw(\mu - \mu_0) H - jw\mu_0 (H - H_0)$$

Introduce $M_{eq} = jw(\mu - \mu_0) H$. So:

$$\left[\nabla \times E_s = -M_{eq} - jw\mu_0 H_s\right]$$

Looks like fields in vaccuum responding to a magnetic current.

Similarly, we can get:

$$\left[\nabla \times H_s = J_{eq} + jw\varepsilon_0 E_s\right]$$
where $J_{eq} = jw(\varepsilon - \varepsilon_0) E$

Ask: where is J_{eq}, M_{eq} non zero?

So: