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Advanced theorems & Equivalences in electromagnetics.

[Bridge from what we've done so far to this topic:
So far we have got the recipe for going from $\bar{J}, \bar{M}$ to $\bar{E}, \bar{H}$ (via $\vec{A}, \vec{F}$). However many antenna problems involve apertures which radiate, e.g. horn antennas or slot antennas or reflect array antennas. In these we can estimate the fields on the aperture, but how do we go from these fields to far fields? So far we only knew how to go from $J, M \Rightarrow$ Equivalence thms! Here we will finally use the unphysical "$\vec{M}$"]

① Duality theorem.

Basic idea: when 2 eqns are of the same form their soln will also be the same.

e.g. if $\frac{d^2 f}{dx^2} + k^2 f = 0$ has solns $f_1(x)$ & $f_2(x)$

for given boundary condns then $\frac{d^2 h}{dx^2} + k^2 h = 0$ will also have solns $h = f_1(x)$ & $h = f_2(x)$ for the same b.c. [super obvious].

How do we make this interesting? Say that we know the soln to $J \neq 0, M = 0$. Can we get the soln to $J = 0, M \neq 0$ from this? i.e. without resolving? Let's see:

Procedure: List the known eqns in the left column, and then the most symmetric looking counterparts in here

	Electric sources $J \neq 0,\ M = 0$	Magnetic sources $J = 0,\ M \neq 0$
1)	$\nabla \times E_A = -j\omega\mu \bar{H}_A$	$\nabla \times \bar{H}_F = j\omega\varepsilon \bar{E}_F$
2)	$\nabla \times \bar{H}_A = \bar{J} + j\omega\varepsilon \bar{E}_A$	$\nabla \times \bar{E}_F = -\bar{M} - j\omega\mu \bar{H}_F$
3)	$\nabla^2 \bar{A} + k^2 \bar{A} = -\mu \bar{J}$	$\nabla^2 F + k^2 F = -\varepsilon M$
4)	$\bar{A} = \int J \otimes g\, dv$	$F = \int M \otimes g\, dv$
5)	$\bar{H}_A = 1/\mu\ \nabla \times \bar{A}$	$E_F = -1/\varepsilon\ \nabla \times F$
6)	$\bar{E}_A = -j\omega \bar{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot A)$	$H_F = -j\omega F - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot F)$

↳ Now ask, if I know the soln for the left, how do I interpret the soln for the right? These are the substitutions:

for ① $E_A \to H_F,\ H_A \to -E_F,\ \mu \to \varepsilon$ ✓

for ② $\underbrace{H_A \to -E_F,\ E_A \to H_F}_{\text{(already done)}},\ J \to M$ ✓

for ③ $A \to F,\ \underbrace{\mu \to \varepsilon,\ J \to M}$ ✓

for ④ $\underbrace{A \to F,\ J \to M}$ ✓

for (5) $\quad \underbrace{H_A \longrightarrow -E_F \,,\; \mu \rightarrow \varepsilon}$ ✓

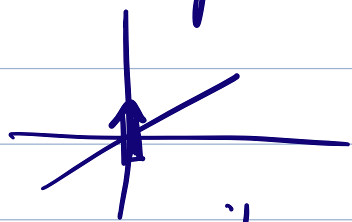
for (6) $\quad \underbrace{E_A \rightarrow H_F \,,\; A \rightarrow F}$ ✓

These eqns did not have η. But $\eta = \sqrt{\mu/\varepsilon}$

$\therefore$ if η appeared, then $\eta \rightarrow \frac{1}{\eta}$.

But $k = \omega^2 \mu \varepsilon$? $k \rightarrow k$.

↳ Very simple example: Hertz dipole is given by a $\vec{J}$:



In the far field

$$E_\theta = j\eta \frac{I_0 l \sin\theta}{4\pi r} e^{-jkr}, \quad H_\phi = jk\frac{I_0 l \sin\theta}{4\pi r} e^{-jkr}$$

$E_r, E_\phi = 0 \qquad\qquad H_r, H_\theta = 0$

Now instead of $\vec{J}$, lets assume $\vec{M}$

i.e. $\vec{M}(r) = \begin{cases} I_m \hat{z}, & |z| < l/2 \\ 0, & \text{else} \end{cases}$

This is apply duality from $\vec{J}$ to $\vec{M}$

(also, this is not impractical, a small loop of electric current is like a $\hat{z}$ directed $\vec{M}$)

(upcoming HW problem)

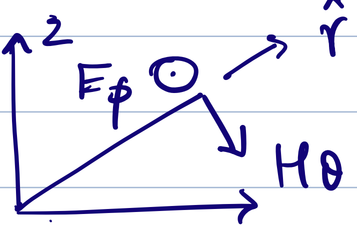
From our recipe, how to write soln?

$E \rightarrow H \,,\; H \rightarrow -E \,,\; J \rightarrow M \,,\; \eta \rightarrow 1/\eta$

$$\therefore \quad H_\theta = j\frac{I_m l \sin\theta}{\eta\, 4\pi r} e^{-jkr}, \quad E_\phi = -jk\frac{I_m l \sin\theta}{4\pi r} e^{-jkr}$$

$H_r, H_\phi = 0 \qquad\qquad E_r, E_\theta = 0.$

Is it still an outgoing TEM wave?



Yes $-\hat{\phi} \times \hat{\theta} = \hat{r}$

E H

Earlier $\hat{\theta} \times \hat{\phi} = \hat{r}$

E H

② Uniqueness Theorem. Statement:

Consider a medium with some nonzero loss (however small), and some sources J_i, M_i (impressed). Then the fields created are unique if any one holds:

ⓐ E_{tan} specified over the bdry

ⓑ H_{tan} " " " "

ⓒ E_{tan} specified over one part & H_{tan} over another.

Proof: straightfwd. Assume J_i, M_i produce two sets of fields $\to \{E_a, H_a\}$ and $\{E_b, H_b\}$. Now when you apply energy conservation thm (Poynting's thm) you can show that under either of a, b, c conditions, $\delta E = E_a - E_b$ and $\delta H = H_a - H_b$ are both 0. [See see 7.3 of AE]. There we need loss.