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→ Now that we have Green's fn we can get back to the antenna problem. (lect 05)

Recall: $\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$ and $\vec{J} = J_z \hat{z}$

& we wanted to solve $\nabla^2 A_z + k^2 A_z = -\mu J_z(r)$

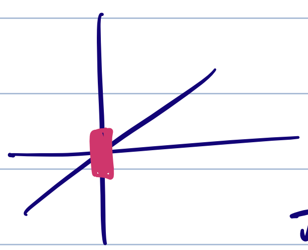
We now know $A_z(r) = \int_{-\infty}^{\infty} d^3r' \, g(r,r') \left(\mu J_z(r')\right)$

Similarly when solving $\nabla^2 \vec{F} + k^2 F = -\varepsilon \vec{M}$

we will get $\vec{F}(r) = \int_{\vec{\infty}} d^3r' g(r,r') (\varepsilon \vec{M}(r'))$

↳ Now let's study an imp special case, the Hertz dipole, a.k.a simplest antenna. [Insights from here extend to all antennas]

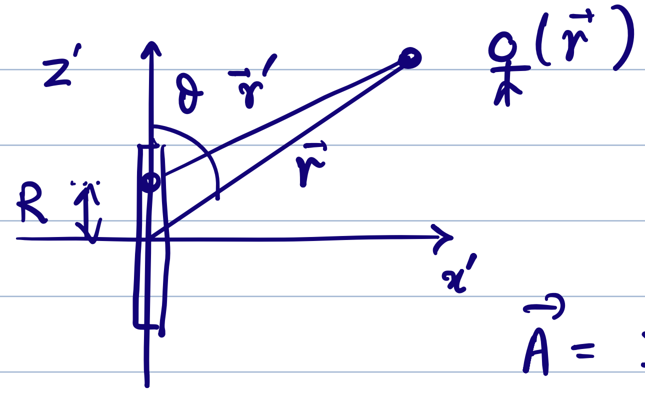
Assume : $J_z(r) = \begin{cases} I_0 \delta(x)\delta(y)\hat{z}, & |z| < l \\ 0, & \text{else} \end{cases}$



further assume $l \ll \lambda$. $\rightarrow \left(\frac{2\pi}{\lambda}\right)$

Soln: $A_z(r) = \int_{-l/2}^{l/2} \frac{I_0 \mu e^{-jk|r'-r|}}{4\pi |r'-r|} dz'$

Recall the geometry for this integral.



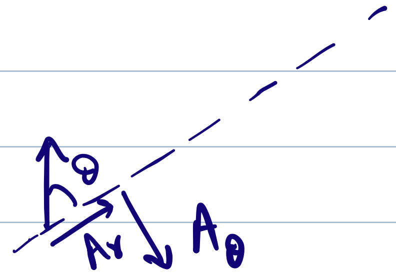
Since $|r'| << \lambda$

This becomes :

$$\vec{A} = \frac{I_0 \mu l\, e^{-jkr}}{4\pi r}\hat{z}$$

Makes more sense to full put it into (r, θ, ϕ)

$$\therefore A_r = A_z \cos\theta = \mu \frac{I_0 l}{4\pi r} e^{-jkr} \cos\theta$$



$$A_\theta = -A_z \sin\theta$$
$$= -\mu \frac{I_0 l}{4\pi r} \sin\theta\, e^{-jkr}$$

$$A_\phi = 0$$

Now let's get the fields: $\bar{H} = \frac{1}{\mu} \nabla \times \bar{A}$

$$= \frac{1}{\mu}\hat{\phi}\left[\frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right)\right]$$

Substituting we get, $H_\phi = j \frac{k I_0 l \sin\theta}{4\pi r}\left(1 + \frac{1}{jkr}\right) e^{-jkr}$

Remember $\vec{E} = \frac{1}{j\omega\varepsilon} \nabla \times \bar{H}$

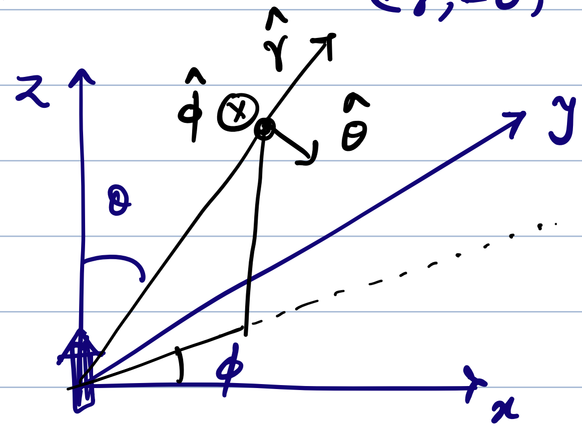
Using $\nabla \times$ in spherical coordinates.

$$E_r = \eta \frac{I_0 l \cos\theta}{2\pi r^2}\left(1 + \frac{1}{jkr}\right) e^{-jkr}$$

$$E_\theta = j\eta \frac{k I_0 l \sin\theta}{4\pi r}\left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) e^{-jkr}$$

$$E_\phi = 0.$$

So, we have: $(E_r, E_\theta, 0)$ & $(0, 0, H_\phi)$



Is this TEM?

Can't tell right away, need to see where the Poynting vector is!

$$S = \frac{1}{2} E \times H^* \quad \text{(complex Poynting vector)}$$

$$= \frac{1}{2}\left(\hat{r} E_r + \hat{\theta} E_\theta\right) \times \left(\hat{\phi} H_\phi^*\right)$$

$$S = \frac{1}{2}\left(\hat{r} E_\theta H_\phi^* - \hat{\theta} E_r H_\phi^*\right)$$

Substituting: $$S_r = \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}\left(1 - j\frac{1}{(kr)^3}\right)$$

$$S_\theta = j\eta \frac{k|I_0 l|^2 \cos\theta \sin\theta}{16\pi^2 r^3}\left(1 + \frac{1}{(kr)^2}\right)$$

Interesting what happens when we integrate this power over an arbitrary sphere.

$$P = \iint \vec{S} \cdot \vec{ds} = \int_0^{2\pi}\int_0^{\pi}\left(\hat{r} S_r + \hat{\theta} S_\theta\right) \cdot \hat{r} r^2 \sin\theta \, d\theta \, d\phi$$

nothing

Gives: $P = \int_0^{2\pi}\int_0^{\pi} S_r \, r^2 \sin\theta \, d\theta \, d\phi \; \hat{r}$

No approx here about r being small or large!

Time avg

$= \hat{r}\, \eta \frac{\pi}{3} \left| \frac{I_0 l}{\lambda} \right|^2 \left[1 - j \frac{1}{(kr)^3} \right]$

Power

$P_{rad} = Re[P]$

$= \eta \frac{\pi}{3} \left| \frac{I_0 l}{\lambda} \right|^2 \equiv \frac{1}{2} |I_0|^2 R_r$

R_r is an equivalent qty, called the radiation resistance of the antenna.

↳ For this dipole antenna, $R_r \simeq 80\pi^2 \left(\frac{l}{\lambda}\right)^2$

if $l = \frac{\lambda}{50}$, $R_r \simeq 0.32\ \Omega$

Terrible antenna, since far from $50\ \Omega$ matching line! HW: what is Γ?

↳ Let's get back to the fields.

Field terms vary as $\underbrace{\frac{1}{r}}$, $\underbrace{\frac{1}{r^2}}$, $\underbrace{\frac{1}{r^3}}$

$\frac{1}{r}$: far field or radiation field (only term that survives far away)

$\frac{1}{r^2}$: Inductive field (Biot Savart law)

$\frac{1}{r^3}$: electrostatics field, (dipole charge like eqn.)

	Near field $kr \ll 1$	Far field $kr \gg 1$
E_r	$\propto j/r^3$	0 (relatively)
E_θ	$\propto j/r^3$	j/r
H_ϕ	$\propto 1/r^2$	j/r
S	$\propto \frac{1}{r^2}, \frac{j}{r^3}, j/r^5$	$1/r^2$
S_{av}	$\propto 1/r^2\ \hat{r}$ / 0 OR (depending on your view)	$1/r^2\ \hat{r}$

$\Rightarrow$ TEM wave in far field.

Rough approximate boundary is $r_0 = \frac{2D^2}{\lambda}$

where D is max dimension of the antenna.

— x —

We saw a specific case of an antenna derivation (dipole with specific current distrib).

So, in general for a $\vec{J}$ in a bounded region, what all can we say?

1) From $\vec{J} \rightarrow$ we get $\vec{A}$ using g.

2) form of $\vec{A}$? $\propto \frac{1}{r^n}$, $n = 1$ far field, $n > 1$ near field (Saw for Hertz)

i.e. $\vec{A} \approx \left[\hat{r}\, A_r'(\theta,\phi) + \hat{\theta}\, A_\theta'(\theta,\phi), \hat{\phi}\, A_\phi'(\theta,\phi)\right] \dfrac{e^{-jkr}}{r}$

Most antenna problems, we are interested in the far zone behaviour. ← (for large r)

Like we saw in the dipole case, $A_r'(\theta,\phi) \approx 0$. $\left[\text{Has } \dfrac{1}{r^n}, n>1\right]$

Subst. $\vec{A}$ into $\vec{E} = -j\omega\vec{A} - j\dfrac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{A})$

Again, under far field approx, this turns out to be:

$$\bar{E}_A = -j\omega\vec{A} \Rightarrow E_r = 0$$

$$E_\theta \approx -j\omega A_\theta$$

$$E_\phi \approx -j\omega A_\phi$$

$$\left.\begin{aligned} H_r &\approx 0 \\ H_\theta &\approx \frac{j\omega A_\phi}{\eta} = -\frac{E_\phi}{\eta} \\ H_\phi &\approx \frac{-j\omega A_\theta}{\eta} = \frac{E_\theta}{\eta} \end{aligned}\right\} \quad H_A \approx \frac{\hat{r}\times\bar{E}_A}{\eta} \Rightarrow \underline{\text{TEM wave}}.$$

[Q: How many pols hidden here ?]

Similarly we can write for $H_F = -j\omega\vec{F}$ & so on.

The above expressions are powerful as they apply to all bounded source (in space) antennas in the far field.

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