Review of wave propagation

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① Maxwell's eqns → without currents

↓ with currents ↓ ② propagation & scattering

↓ ③ Antennas & arrays.

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① Start with Maxwell's eqns.

(Faraday) (Ampere)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_f$$

$$\nabla \cdot \vec{D} = \rho_f, \quad \nabla \cdot \vec{B} = 0$$

(Gauss)

M.E.

Constitutive relns $\bar{B} = \mu \bar{H}$, $\bar{D} = \varepsilon \bar{E}$

μ, ε can be tensors, can be fns of frequency.

Take home msg: space varying fields ⟺ time varying fields

↳ Specialize to EE usage & formulation.

① assume all fields of the form $\left[e^{j\omega t} \times fn(\text{space})\right]$

$\Rightarrow \frac{\partial}{\partial t} \rightarrow j\omega$

② Use E, H instead of D, B.

Leads to:

$$\nabla \times \vec{E} = -j\omega\mu H, \quad \nabla \times \bar{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

i.e. 2 eqns in 2 variables. Eliminate one:

$\nabla \times (\nabla \times E) = -j\omega\mu [\vec{J} + j\omega\varepsilon \vec{E}]$

$$\nabla(\nabla \cdot E) - \nabla^2 \bar{E} = -j\omega\mu \vec{J} + \omega^2 \mu \varepsilon \bar{E}$$, rearrange

$$\nabla^2 E + \omega^2 \mu \bar{E} - \nabla(\nabla \cdot \bar{E}) = j\omega\mu \vec{J}$$

If we assume linear, charge free medium, $D = \varepsilon E$
$\Rightarrow \nabla \cdot D = \rho_f = \varepsilon \nabla \cdot E = 0$ and no current,
we get the "wave" eqn:

$\nabla^2 E = -\omega^2 \mu E$. Make it even simpler:
Assume $E(x,y,z) = E(z)$ alone:
$$\Rightarrow \frac{\partial^2 E(z)}{\partial z^2} = -\omega^2 \mu\varepsilon E(z).$$

Simplifications: $\mu\varepsilon = \mu_0 \varepsilon_0 \varepsilon_r$ → relative permittivity
and $c = 1/\sqrt{\mu_0 \varepsilon_0}$, so $\omega^2 \mu \varepsilon = \frac{\omega^2}{c^2} \varepsilon_r$

Now $\omega = 2\pi f$ and $f/c = 1/\lambda_0 \Rightarrow \omega/c = 2\pi f/\lambda_0$

This $2\pi/\lambda_0 = k_0$ [wave vector]

Finally $$\frac{d^2 E}{dz^2} = -(k_0^2 \varepsilon_r) E$$

Solns? 2nd order DE ⇒ 2 indep solns

$$E = E_0 e^{\pm j k_0 \sqrt{\varepsilon_r} z} \quad \text{or} \quad \left\{ \begin{matrix} \sin(k_0 \sqrt{\varepsilon_r} z) \\ \cos(k_0 \sqrt{\varepsilon_r} z) \end{matrix} \right\}$$

↳ Let's make this more physical by bringing back time; $e^{j\omega t}$ and assume free space ($\sqrt{\epsilon_r} = 1$).

$\Rightarrow E_1 = e^{j(\omega t - k_0 z)}$, $E_2 = e^{j(\omega t + k_0 z)}$

what to they mean?

→ <u>Phase front</u>: locus of points in space-time with constant phase.

For E_1? when $t = \frac{k_0}{\omega} z$, const phase

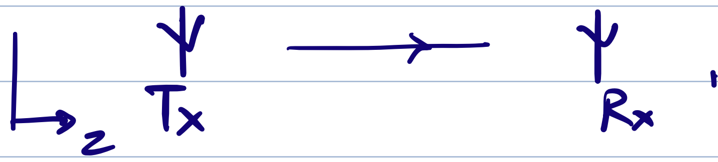
$\Rightarrow$ as $t \uparrow$, z also $\uparrow$

$\Rightarrow$ phase front moves along $+z$ as $t \uparrow$.
Called the fwd travelling wave.

Similarly E_2 phase front? $t = -\frac{k_0}{\omega} z$

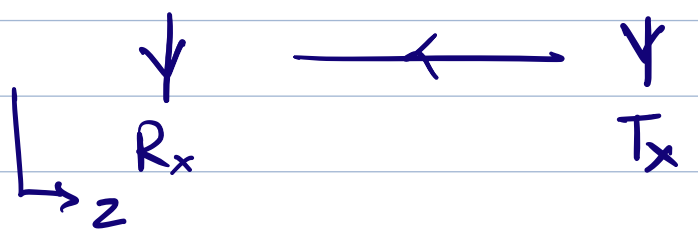
$\Rightarrow$ backward travelling wave.

Why is this imp? If like so:



, wave at Rx $\propto e^{j(\omega t - k_0 z)}$

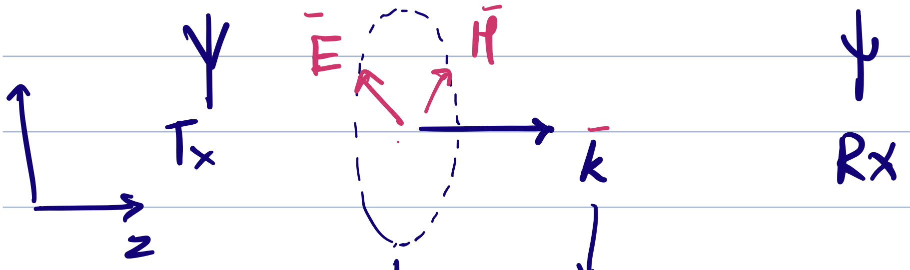
But in this situation:



, wave at Rx $\propto e^{j(\omega t + k_0 z)}$

Simple concept, but imp to be clear.
Sometimes ppl drop $e^{j\omega t}$. $e^{\pm j k_0 z} \rightarrow$ <u>phasor notation</u>.

Let's put some more directions.



dir of propagation

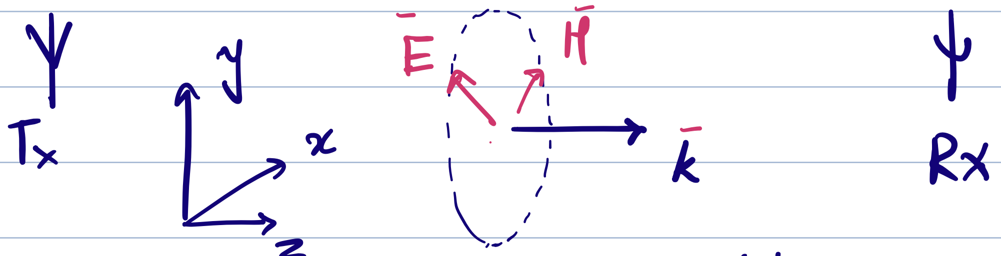
plane $\perp^r$ to this

E & H live in this plane: $\{\vec{E}, \vec{H}, \vec{k}\}$ form a right handed coord sys.

$$\hat{E} \times \hat{H} = \hat{k}.$$

Polarization: Locus of the tip of $\vec{E}$ vector in time/space.

Practically: Stand at a fixed point in space and see what the E field does.



eg. $\vec{E}$ can be $(a_x e^{-jkz}, 0, 0)$ or

E can be $(0, a_y e^{j\delta} e^{-jkz})$ $[a_x, a_y \in \mathbb{R}]$

or superposition.

$$(a_x, e^{j\delta} a_y, 0) e^{-jkz}.$$

(at fixed z)

Since pol is a locus in time we have

to go from phasor domain to time domain.

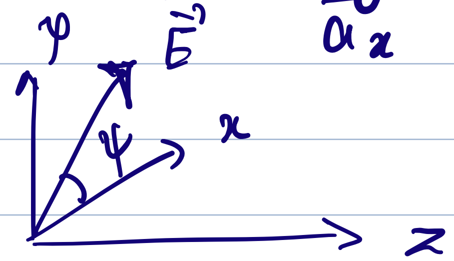
Standard recipie:

$$\vec{field}(r,t) = Re\left[\vec{field}(r) \times e^{j\omega t}\right]$$

$$\Rightarrow \vec{e}(z,t) = \left(a_x \cos(\omega t - kz),\ a_y \cos(\omega t - kz + \delta),\ 0\right)$$

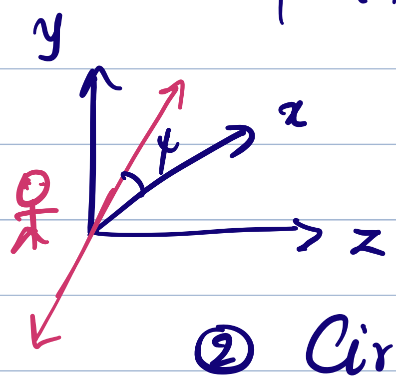
Mag $|\vec{e}(z,t)| = \sqrt{a_x^2 \cos^2(\ ) + a_y^2 \cos^2(\ )}$

dir $\tan\psi = \dfrac{a_y}{a_x} \dfrac{\cos(\omega t - kz + \delta)}{\cos(\omega t - kz}$



Cases ① Linear pol $\Rightarrow \delta = 0 \Rightarrow \psi$ fixed in t.

$$\Rightarrow |\vec{e}(z,t)| = \sqrt{a_x^2 + a_y^2}\cos(\omega t - kz)$$



$$\tan\psi = a_y / a_x$$

② Circ pol $\Rightarrow \delta = \pm\pi/2$, $a_x = a_y = a$

$|\bar{e}(z,t)| = a$ (const in t)

$\psi = \pm(\omega t - kz_0)$

$\delta = -\pi/2$

$\delta = +\pi/2$

This gives Right or left circular pol

③ Anything else is Elliptical pol.

Attenuation. How do we formulate wave prop in a medium which has loss? eg: mm-wave in rain or substrate loss in FR4?

All boils down to $\boxed{\varepsilon_r}$.

Recall $\dfrac{d^2E(z)}{dz^2} = -k_o^2 \, \varepsilon_r \, E(z)$

Now write $\varepsilon = \varepsilon' - j\varepsilon''$ loss

↳ sometimes written like $\underset{\text{(complex pertivity)}}{\varepsilon_c} = \underset{\text{(rel perm)}}{\tilde{\varepsilon}_r}\varepsilon_o(1 - j \underset{\text{(Loss tangent)}}{\underbrace{\tan\delta}})$ in data sheets.

$$\Rightarrow \quad k = k_o\sqrt{\varepsilon_r} = k_o\sqrt{\tilde{\varepsilon}_r(1-j\tan\delta)} = k_r - jk_i$$

$$e^{-jkz} = e^{-jk_r} \, e^{-k_i z}$$

Full picture: $e^{j(\omega t - kz)} = \underbrace{e^{-k_i z}}_{\text{decays as it propagates}} \, \underbrace{e^{j(\omega t - k_r z)}}_{\text{usual propagation.}}$

{skin depth $1/k_i$} ← {decays as it propagates}

★ different from path loss → will cover this later.

Note: keep signs in mind. For a wave going ←

$$E \propto e^{+k_i z} \, e^{j(\omega t + k_r z)} \quad (\text{physical} \checkmark)$$

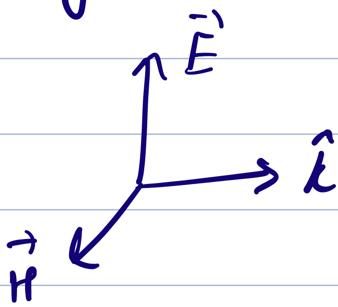
↳ Few more facts about EM wave propagation

① Relation between $\bar{E}$ & $\bar{H}$ in homogeneous mediums: $\dfrac{|\bar{E}|}{|\bar{H}|} = \eta$ (characteristic imp of free space)

In vacuum, $\eta_0 = 120\pi \approx 377\,\Omega$.

[More generally $\vec{H} = -\frac{1}{\mu} \nabla \times \bar{E}$]

② Power.

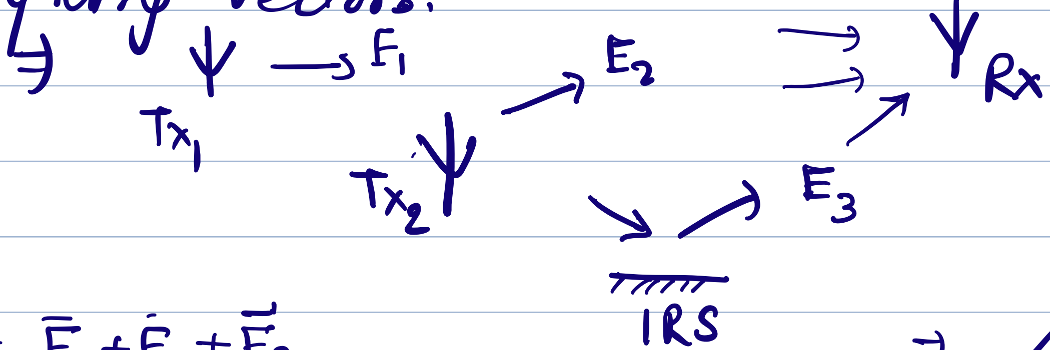


Instantaneous power, $S(\bar{r},t) = \vec{e}(r,t) \times \vec{h}(r,t)$
Poynting vector $\underbrace{\quad}_{W/m^2}$

More practical: time averaged qty is

$$\bar{S}_{AV} = \frac{1}{2} \mathrm{Re}\left[E \times H^* \right] \quad W/m^2 \quad \leftarrow \text{phasors}$$

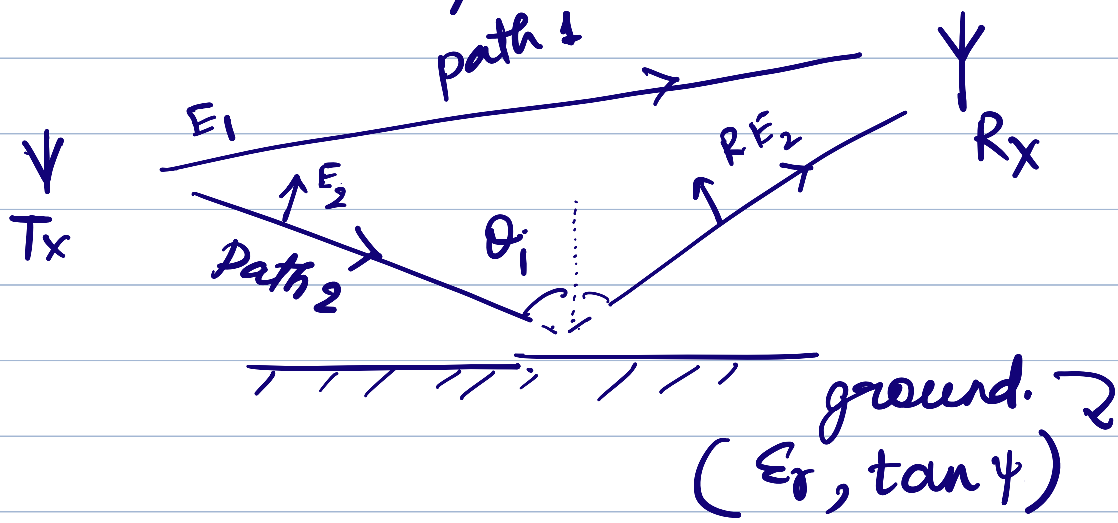
③ Superposition theorem applies to fields, not Poynting vectors.



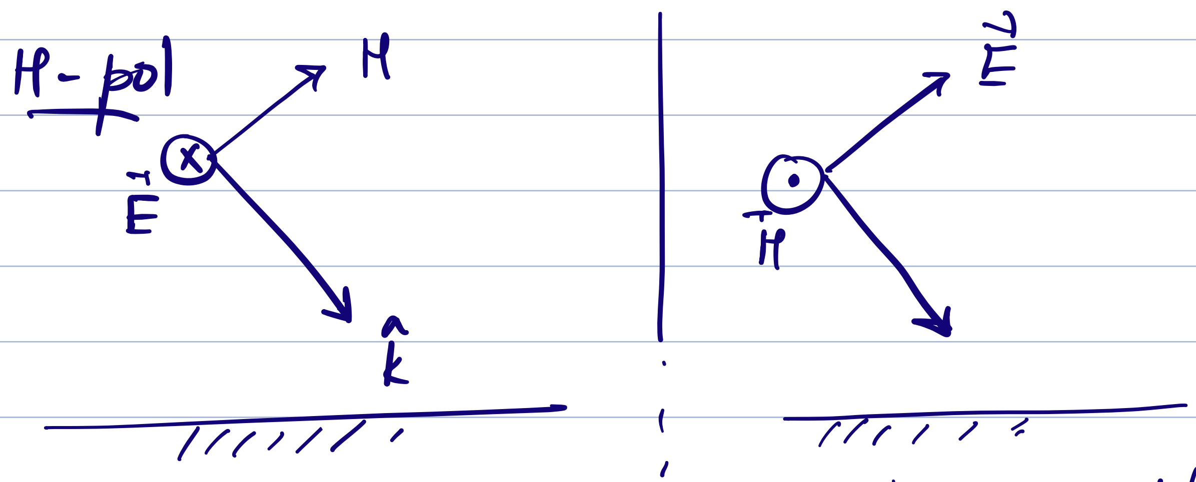
⇒ Net $E = \bar{E}_1 + \bar{E}_2 + \vec{E}_3$ ↘ from here $\rightarrow \vec{S}$ ✓

Not $S = \bar{S}_1 + \bar{S}_2 + \bar{S}_3$ ✗

④ How to setup a multipath env.?
Need to know polarization & materials.



Different refl coefs for different pols
2 formulae for H & V-pol. (linear pols)
So $R_H(\theta_i, \varepsilon_g)$ & $R_V(\theta_i, \varepsilon_g)$



$\bar{E}$ is parallel to ground.

E is not parallel to ground

① Combine all the fields by superposn

② Usually $R_H \neq R_V \Rightarrow$ No longer CP after reflection.

③ $R_H, R_V = \pm 1$ for metals.