From Maxwells equations to a simple electromagnetic wave

1) Time Harmonic Wave Eqn: $\nabla^2 \vec{E}(x,y,z) = -\omega^2 \mu \varepsilon \vec{E}(x,y,z)$
It is also valid for each component of $\vec{E}$ and $\vec{H}$

2) w.l.o.g. assume $\vec{E}$ along $\hat{x}$ direction. Thus $\vec{E} = (E_x(x,y,z), 0, 0)$

3) $\vec{E}$ satisfies $\nabla \times \vec{E} = -j\omega\mu \bar{H} \Rightarrow \left(0, \frac{\partial}{\partial z} E_x, -\frac{\partial E_x}{\partial y}\right)$
$$= -j\omega\mu (H_x, H_y, H_z) \quad -(1)$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix}$$

<u>Obsv #1</u> $H_x = 0$.

4) We wrote E_x as $E_x(x,y,z)$. Several possibilities exist:
a) E_x = const. If so, then $\vec{H} = 0$ from (1), but then $\nabla \times \bar{H} = j\omega\varepsilon_0 \vec{E}$ gives $\vec{E} = 0$ } Contradiction
b) $E_x = E_x(x)$. Same as above $\Rightarrow \vec{H} = 0$ }
c) $E_x = E_x(y)$. Leads to $\vec{H} = (0, 0, H_z)$
d) $E_x = E_x(z)$. Leads to $\vec{H} = (0, H_y, 0)$
e) $E_x = E_x(y,z)$. Leads to $\vec{H} = (0, H_y, H_z)$
f) $E_x = E_x(x,y,z)$. " " " "
} No seeming contradiction. (so far)

5) To further narrow the possibilities, consider the eqn:
$\nabla \times \vec{H} = j\omega\varepsilon_0 \vec{E}$ for the cases (c,d,e,f) above,
i.e. for $\bar{H} = (0, H_y, H_z)$.
$$\nabla \times \vec{H} = \left(\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial y} H_y, \underbrace{-\frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x}}\right) = j\omega\varepsilon_0 (E_x, \underbrace{0, 0})$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & H_y & H_z \end{vmatrix}$$

Must be consistent

$\Rightarrow$ Both $\frac{\partial H_z}{\partial x} = 0$ and $\frac{\partial H_y}{\partial x} = 0$ and so $\vec{H}$ is only a fn of (y,z) NOT x.
<u>obsv #2</u>

6) Going back to eqn (1), we get: $H_y(y,z) = \frac{j}{\omega\mu} \frac{\partial E_x}{\partial z}$ and
$$H_z(y,z) = \frac{-j}{\omega\mu} \frac{\partial E_x}{\partial y}$$

Since the LHS are only fns of (y,z), RHS must also be so

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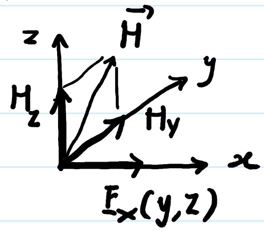
$\Rightarrow \; E_x = E_x(y,z)$ <u>obsv # 3</u>.

Thus possibility (f) in point (4) is invalid.

7) The fields we have now are:



Thus even in the most general case $\vec{E}$ and $\vec{H}$ are perpendicular. ← <u>obsv # 4</u>

8) In this general setting we can make life even easier by aligning our coordinate axis in such a way that $\vec{E}$ is along $\hat{x}$ and $\vec{H}$ along $\hat{y}$. Our choice!

So $\vec{E} = (E_x, 0, 0)$ and $\vec{H} = (0, H_y, 0)$.

9) Above $\Rightarrow H_z = 0$. From eqn (1) we then get $H_z \propto \frac{\partial}{\partial y} E_x = 0$

This suggests E_x is not a fn of y, only z.

So we have narrowed it down to case (d) from pt (4).

The same eqn gave $H_y \propto \frac{\partial E_x}{\partial z} \Rightarrow H_y$ is only a fn of z.

10) <u>Conclusion</u>: Even the most general situation can be written as $\vec{E} = (E_x(z), 0, 0)$ and $\vec{H} = (0, H_y(z), 0)$ by convenient choice of coordinate system.