EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 14: Introduction to Transformation of Random Variables

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Suppose we are able to observe a random variable or a collection of random variables. In many practical situations, we may be more interested in some *function* of the observed random variable(s). For example, in communication systems, the logarithm of the noise power is often more useful to an engineer than the noise realisation itself.

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. We are interested in characterising the properties of $f(X)$. Since random variable X is itself a function, $f(X)$ is a composed function that maps Ω to $\mathbb{R}$. First, we have to ask if $f(X)$ is indeed a legitimate random variable. Consider the composed function $f \circ X(\cdot)$, depicted in Figure . If f is an arbitrary function, $f(X)$ may not be a random variable. However, if $f : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function (i.e., pre-images of Borel sets under f are also Borel sets), then it is clear that the pre-images of Borel sets under the composed function $f \circ X(\cdot)$ are events (why?), and it follows that $f(X)$ is indeed a random variable. Similarly, for a Borel-measurable function $f : \mathbb{R}^n \to \mathbb{R}$, and random variables $X_1, X_2, X_3, \ldots, X_n$, it can be argued that $f(X_1, \ldots, X_n)$ is a random variable.

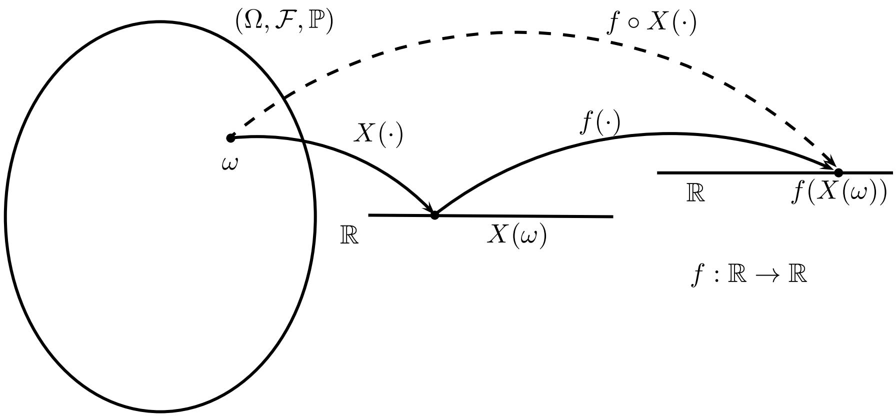

Figure 14.1: Transformation of random variable

Now that we have established conditions under which a function of a random variable is a random variable, we ask after the probability law of $f(X)$, given the probability law $\mathbb{P}_X$ of X. Equivalently, given the CDF of X, we want to find the CDF of $f(X)$. We begin by considering some elementary functions such as maximum, minimum, and summations, and then proceed to more general transformations.

14.1 Maximum and Minimum

Let $X_1, X_2, X_3, \ldots, X_n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with joint CDF $F_{X_1, X_2, \ldots, X_n}$. Define

$$Y_n = \min(X_1, X_2, X_3, \ldots, X_n).$$

and

$$Z_n = \max(X_1, X_2, X_3, \ldots, X_n).$$

Here we are interested in finding the CDF of Y_n and Z_n.

First let us check that Z_n is indeed a random variable. Note that $\{Z_n \leq x\}$ is equivalent to saying that each of $X_1, X_2, X_3, \ldots, X_n$ is less than or equal to x. Thus, we have,

$$\{Z_n \leq x\} = \{X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x\}.$$

Now, in order to see that $\{\omega : Z(\omega) \leq z\}$ is an event, note that $\{\omega : Z(\omega) \leq z\} = \bigcap_{i=1}^{n} \{\omega : X_i(\omega \leq z)\}$. This is a finite intersection of events, since the X_is are random variables. Therefore, Z_n is a legitimate random variable.

Next, for the minimum, note that the if $\{Y_n > x\}$ is equivalent to saying that each of $X_1, X_2, X_3, \ldots, X_n$ is greater than x. Thus,

$$\{Y_n > x\} = \{X_1 > x, X_2 > x, \ldots, X_n > x\}.$$

We can prove that Y_n is also a random variable, by using arguments similar to those used for proving that Z_n is a random variable.

We now proceed to compute the CDF of random variables, Z_n and Y_n.

$$\begin{aligned}
\mathbb{P}(\{Z_n \leq x\}) &= \mathbb{P}(\{X_1 \leq x\} \cap \{X_2 \leq x\} \cdots \cap \{X_n \leq x\}), \\
&= F_{X_1, X_2, \ldots, X_n}(x, x, \ldots, x).
\end{aligned}$$

Similarly for Y_n,

$$\begin{aligned}
\mathbb{P}(\{Y_n > x\}) &= \mathbb{P}(\{X_1 > x\} \cap \{X_2 > x\} \cdots \cap \{X_n > x\}\}), \\
\bar{F}_{Y_n}(x) &= 1 - F_{Y_n}(x), \\
&= \bar{F}_{X_1, X_2 \ldots. X_n}(x, x, \ldots, x),
\end{aligned}$$

where $\bar{F}_{X_1 X_2, \ldots, X_n}(\cdot)$ denotes the joint complementary CDF.

In particular if $X_1, X_2, \ldots, X_n$ are independent

$$\begin{aligned}
F_{Z_n}(x) &= F_{X_1}(x) F_{X_2}(x) \ldots F_{X_n}(x). \\
\bar{F}_{Y_n}(x) &= \bar{F}_{X_1}(x) \bar{F}_{X_2}(x) \ldots \bar{F}_{X_n}(x).
\end{aligned}$$

Further if they are i.i.d (independent and identically distributed), then

$$\begin{aligned}
F_{Z_n}(x) &= [F_X(x)]^n. \\
\bar{F}_{Y_n}(x) &= [\bar{F}_X(x)]^n.
\end{aligned}$$

Example 1:- Consider U_1, U_2 to be i.i.d, Unif$[0, 1]$,
Let Y $= \min(U_1, U_2)$ and $Z = \max(U_1, U_2)$.
Let $F_{U_1}(z)$ and $F_{U_2}(z)$ be CDF's of random variables U_1 and U_2 respectively. Since they are identically distributed

$$F_{U_1}(z) = F_{U_2}(z) = [F_U(z)],$$

where

$$F_U(z) = \begin{cases} 0 & z < 0, \\ z & z \in [0,1], \\ 1 & z > 1. \end{cases}$$

Since U_1 and U_2 are also independent

$$F_Z(z) = F_{U_1}(z)F_{U_2}(z) = [F_U(z)]^2.$$

$$[F_U(z)]^2 = \begin{cases} 0 & z < 0, \\ z^2 & z \in [0,1], \\ 1 & z > 1. \end{cases}$$

Its pdf is given by

$$f_z(z) = \begin{cases} 2z & z \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for $F_Y(y)$ we can write

$$\bar{F}_Y(y) = \bar{F}_{U_1}(z)\bar{F}_{U_2}(z) = [\bar{F}_U(y)]^2,$$

where $\bar{F}_Y(y)$ denotes the complementary CDF of Y.

$$F_Y(y) = 1 - [\bar{F}_U(y)]^2.$$

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ 1-(1-y)^2 & y \in [0,1], \\ 1 & y > 1. \end{cases}$$

The pdf is given by

$$f_y(y) = \begin{cases} 0 & y < 0, \\ 2(1-y) & y \in [0,1], \\ 1 & y > 1. \end{cases}$$

Example 2:-

Let $X_1, X_2, X_3, \ldots, X_n$ be independent random variables which are exponentially distributed with the parameters $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n > 0$. $F_{X_i}(x) = 1 - e^{-\lambda_i x}$ for $x > 0$.

Let

$$Y_n = \min(X_1, X_2, ...X_n).$$

Then the complementary CDF of Y_n :

$$\begin{aligned}\bar{F}_{Y_n}(y) &= \prod_{i=1}^{n} \bar{F}_{X_i}(y), \\ &= \prod_{i=1}^{n} e^{-\lambda_i y}, \\ &= e^{(-\sum\limits_{i=1}^{n} \lambda_i) y}.\end{aligned}$$

We can see that Y_n is an exponential random variable with parameter $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$. Thus, the minimum of independent exponential random variables is another exponential random variable!

14.2 Exercises

1. **Light bulbs with Amnesia:** Suppose that n light bulbs in a room are switched on at the same instant. The life time of each bulb is exponentially distributed with parameter $\mu = 1$, and are independent.
   (a) Starting from the time they are switched on, find the distribution of the time when the first bulb fuses out.
   (b) Find the CDF and the density of the time when the room goes completely dark.
   (c) Would your answers to the above parts change if the bulbs were not switched on at the same time, but instead, turned on at arbitrary times? Assume however that all bulbs were turned on before the first one fused out.
   (d) Suppose you walk into the room and find m bulbs glowing. Starting from the instant of your walking in, what is the distribution of the time it takes until you see a bulb blow out?
2. Let X and Y be independent exponentially distributed random variables with parameters λ and μ respectively.
   (a) Show that $Z = \min(X, Y)$ is independent of the event $\{X < Y\}$, and interpret this result verbally? [Definition: A random variable X is said to be independent of an event A if X and $\mathbb{I}_A$ are independent random variables, where $\mathbb{I}_A$ is the Indicator random variable of the event A.]
   (b) Find $\mathbb{P}(X = Z)$.