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A Linear-Quadratic Game Approach to Estimation and Smoothing

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Abstract

An estimator and smoother for a linear time varying system, over a finite time interval, are developed from a linear quadratic (LQ) game approach. The exogenous inputs composed of the measurement and process noise, and the initial state, are assumed to be finite energy signals whose statistics are unknown. The measure of performance is in the form of a disturbance attenuation function and the optimal estimator (smoother) bounds the attenuation function from above. The disturbance attenuation function is converted to a performance measure for a zero-sum LQ game and the exogenous inputs and the estimator are viewed as players in the game; the exogenous inputs attempt to worsen the estimate while the estimator tries to provide the most accurate estimate. The optimal estimator (smoother), restricted to a class of functions dependent on the measurement alone, is found to be unbiased and linear in structure. With a few mild assumptions, the results are extended to a linear time-invariant system on an infinite horizon, and the optimal estimator obtained is shown to satisfy an upper bound on the H_∞ norm.

1. Introduction

Estimation is the process of finding the best estimate of the states of a dynamic system at a given time, from the information available up to that instant of time. The 'best' is with respect to some measure of performance. Smoothing, improves the estimate by incorporating all the available information, including those available after the estimation time. Estimation is usually implemented to operate on-line, while smoothing is implemented to operate off-line. Both estimation and smoothing are considered here. Throughout this paper, reference to H_∞ and H_2 norms and minimization pertains to transfer function matrices of causal linear time-invariant systems. The claims and results, in this context, are also valid for causal linear time-varying systems; the corresponding notion being the L_2 norms of the signals and the minimization of an attenuation function composed of L_2 functions.

The solution to the estimation and smoothing problem for a linear dynamic system, subject to exogenous signals whose spectral characteristics are known, is well established in the field of control theory [1, 2, 8]. From a mathematical viewpoint, the solution minimizes the covariance of the estimation error, which is equivalent to minimizing the H_2 norm of the transfer function matrix from the external signals to the error. Over the past decade, the attention of control theorists has been focussed on linear systems subject to significant uncertainty in parameters as well as external inputs. This uncertainty has prompted a new measure of performance -the H_∞ norm- which ensures a more robust design. Minimizing the H_∞ norm of the transfer function matrix from the exogenous inputs to the error signal is equivalent to imposing an upper bound on the maximum gain of the error signal over all frequencies. This problem and measure of performance were initially formulated by Zames [3]. In the control problem, the error signal constitutes the difference between the desired output and the actual output. In estimation, the error is the difference between the actual state and the estimate of the state.

Though the control aspect of H_∞ optimization has been studied extensively, significant contributions being [3, 4, 10], the topics of estimation and smoothing in the same setting have received little attention. Doyle *et al* [10] consider a restricted output estimation problem with the constraint of internal stability on the system. Yaesh and Shaked [6] introduce a fictitious signal to derive the state estimator. Doing so, does not provide any physical insight and further, the problem solved is restricted in a sense due to the assumptions on the noise structure and the initial estimate. Nagpal and Khargonekar [7] derive a smoother and estimator from a performance index similar to the one in this paper but most of the proofs there are omitted, and the few provided are not lucid. Much of the previous effort, with the exception of [7], has focussed on linear time-invariant systems on an infinite horizon. Furthermore, the effect of uncertainty in the initial state has not been considered. These factors motivate the need for a solution, and simple formulation, to the H_∞ estimation and smoothing problem. A linear time-varying system on a finite horizon, with uncertainty in the initial state is considered here. Such a system would consist of the most-general case of the problem being addressed. With a mild assumption that the system under consideration is both stabilizable and detectable, the results of the linear time-varying system over a finite time can be extended to linear time-invariant systems on an infinite horizon.

Since H_∞ optimization involves the minimization of the worst possible amplification of the error signal, it can be viewed as a worst-case design and interpreted from a min-max game theoretic approach. In this context, the controller (estimator) is a player in a differential game, prepared for the worst strategy of the adversaries, the exogenous inputs. Petersen's work [12] was one of the first to notice this link between H_∞ optimization and linear-quadratic differential games, through the identical Riccati equations arising in both. If the optimal strategies in a LQ game satisfy a saddle-point condition [8], then this ensures a bound on the H_∞ norm of the transfer function matrix. For a linear time-varying system, a corresponding disturbance attenuation function, used as a measure of performance for the LQ game, is bounded from above [5]. For a short explaination on the relation between H_∞ optimization, LQ games, and time-domain interpretations, the interested reader is referred to the appendix.

This relationship between linear-quadratic differential games and H_∞ theory is exploited in this paper, to formulate the estimation and smoothing problem in a game theoretic setting and in the time domain. The time domain enhances comprehension and the game-theoretic approach is intuitively appealing. The only assumption made here is that the exogenous signals are in L_2 space i.e. signals with bounded energy.

The paper is organized as follows. In section 2, starting with a disturbance attenuation function as a measure of performance, the linear-quadratic game is formulated. In section 3, the optimal strategies are derived, using standard variational techniques, and optimality is established. Based on the optimal strategies obtained in section 3, the optimal estimator is constructed in section 4. In section 5, in a straight forward way the estimation results are extended to determine the optimal smoother. Satisfaction of the bound on the attenuation function and the H_∞ norm is shown in section 6. Finally, necessary and sufficient conditions for optimality are given in section 7.

2. The Disturbance Attenuation Functionand a Differential Game

a. Problem definition

Consider a linear time-varying system, governed by the equation

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{w} \tag{1}$$

and with a linear measurement

$$\mathbf{y} = C\mathbf{x} + D\mathbf{v} \tag{2}$$

over a finite time interval $[0,T]$. $\mathbf{w}(.), \mathbf{v}(.) \in L_2[0,T]$. $\mathbf{w}(t) \in R^m$, $\mathbf{v}(t) \in R^p$, $\mathbf{y}(t) \in R^p$, $\mathbf{x}(t) \in R^n$, $A \in R^{n\times n}$, $B \in R^{n\times n}$, $C \in R^{p\times n}$, $D \in R^{p\times p}$. D is assumed to be invertible, which implies that each of the p measurements is corrupted by noise in p independent directions.

Define the measurement history as $Y_t = (\mathbf{y}(s) : 0 \leq s \leq t)$. The estimate of the state at time t, denoted by $\hat{\mathbf{x}}(t)$, is computed based on the mesurement history up to t. The smoothed estimate of the state at time s, denoted by $\hat{\mathbf{x}}_s(s)$ is computed based on the measurements up to time $t > s$.

Define a vector $\mathbf{z}(t) \in R^q$, which is a linear combination of the states

$$\mathbf{z} = L\mathbf{x} \tag{3}$$

where $L \in R^{q\times n}$. When $L = I^{n\times n}$, the vector $\mathbf{z}$ reduces to the state-vector. The estimate (smoothed estimate) $\hat{\mathbf{z}}^*(\hat{\mathbf{z}}_s)$ of $\mathbf{z}$ belongs to a class of functions $\hat{\mathbf{z}}$ which are piecewise continuous functions of the measurement $\mathbf{y}$.

The measure of performance is in the form of a disturbance attenuation function

$$J_{of} = \frac{\int_0^T \| \mathbf{z} - \hat{\mathbf{z}} \|_Q^2 \, dt}{\| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \int_0^T (\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{v} \|^2_{V^{-1}}) dt} \tag{4}$$

where $(\mathbf{z} - \hat{\mathbf{z}}) \in R^q$, $((\mathbf{x}(0) - \hat{\mathbf{x}}_0), \mathbf{w}, \mathbf{v}) \in (R^n, L_2, L_2) \neq 0$. $\mathbf{z} - \hat{\mathbf{z}}$ is the estimation error, $(\mathbf{x}(0) - \hat{\mathbf{x}}_0)$ is the error in the state-estimate at the initial time $t = 0$, $\hat{\mathbf{x}}_0$ is the initial state-estimate which is known, $\mathbf{w}$ and $\mathbf{v}$ are the process and measurement noise, respectively. Note that for the smoothing case, $\mathbf{z} - \hat{\mathbf{z}}$ is substituted by $\mathbf{z} - \hat{\mathbf{z}}_s$.

$$\int_0^T \| \mathbf{z} - \hat{\mathbf{z}} \|_Q^2 \, dt \equiv \int_0^T (\mathbf{z} - \hat{\mathbf{z}})^T Q (\mathbf{z} - \hat{\mathbf{z}}) dt$$

is the square of the weighted (weighted by Q) L_2 norm of the estimation error $(\mathbf{z} - \hat{\mathbf{z}})$. Similarly, the other terms in the attenuation function are squares of weighted L_2 norms. Q, W, V are time varying, symmetric matrices. Further, $Q \geq 0$, $W > 0$, $V > 0$ and $P_0^{-1} > 0$.

The optimal estimate $\hat{\mathbf{z}}^*$ among all possible $\hat{\mathbf{z}}$ should satisfy

$$sup \quad J_{of} < \frac{1}{\theta} \tag{5}$$

$\forall$ $((\mathbf{x}(0) - \hat{\mathbf{x}}_0), \mathbf{w}, \mathbf{v}) \in (R^n, L_2, L_2) \neq 0$ where *sup* stands for supremum and $\theta (\theta > 0)$ is a scalar.

The above performance criterion leads to a worst case design. The matrices Q, W, V and P_0^{-1} are left to the choice of the designer and depend on performance requirements. P_0^{-1}, in a sense, reflects the uncertainty in the initial estimate. The denominator of the attenuation function can be interpreted as a weighted ball in (R^n, L_2, L_2), i. e. $\mathbf{B}_{R^n \times L_2 \times L_2} = \{((\mathbf{x}(0) - \hat{\mathbf{x}}_0), \mathbf{w}, \mathbf{v});\ \ \| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}}$

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$+\int_0^T(\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{v} \|^2_{V^{-1}})dt \leq 1\}$.

Consider the system to be time-invariant (A, B, C, D are constant matrices), and let $T \to \infty$. Let W, V and Q be identity matrices. Further, assume that (A, B) is controllable and (C, A) is detectable. Now, if the disturbance attenuation function is bounded above by $\frac{1}{\theta}$, then the H_∞ norm of the transfer function matrix from the external signals to the estimation eror (T_{ed}) is also bounded from above [5] (see appendix). That is

$$J_{af}(\hat{\mathbf{z}}^*, \mathbf{w}, \mathbf{v}, \mathbf{x}(0)) < \frac{1}{\theta} \Rightarrow \| T_{ed} \|_\infty < \gamma \tag{6}$$

where $\mathbf{e} = (\mathbf{z} - \hat{\mathbf{z}}^*)$ is the estimation error, $\mathbf{d}$ represents the exogenous inputs $\mathbf{w}, \mathbf{v}$, and $\gamma(\gamma > 0)$ is a scalar.

b. Problem Formulation

The solution to the estimation and smoothing problem can be obtained by converting the performance criterion(5) to a performance index for a differential game. The problem is formulated as a linear-quadratic differential game in which the estimate $\hat{\mathbf{z}}$ plays against the exogenous inputs $\mathbf{w}, \mathbf{v}$, and the initial state $\mathbf{x}(0)$. The performance criterion is

$$\min_{\hat{\mathbf{z}}} \max_{(\mathbf{w},\mathbf{v},\mathbf{x}(0))} J = -\frac{1}{2\theta} \| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \frac{1}{2}\int_0^T \| \mathbf{z} - \hat{\mathbf{z}} \|^2_Q - \frac{1}{\theta}(\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{v} \|^2_{V^{-1}})dt \tag{7}$$

subject to the constraints (1), (2) and (3). Substituting $\mathbf{v} = D^{-1}(\mathbf{y} - C\mathbf{x})$, $\mathbf{z} = L\mathbf{x}$ and $\hat{\mathbf{z}} = L\hat{\mathbf{x}}$ the performance index (7) can be recast in the form

$$\min_{\hat{\mathbf{x}}} \max_{(\mathbf{y},\mathbf{w},\mathbf{x}(0))} J = -\frac{1}{2\theta} \| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \frac{1}{2}\int_0^T \| \mathbf{x} - \hat{\mathbf{x}} \|^2_{\bar{Q}} - \frac{1}{\theta}(\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{y} - C\mathbf{x} \|^2_{\bar{V}})dt \tag{8}$$

where $\bar{Q} = L^TQL$ and $\bar{V} = D^{-T}V^{-1}D^{-1}$ and subject to constraint (1).

c. Order of Optimization

To obtain a solution to the game problem, a certain order of optimization is followed. This order of optimization is critical since the results are dependant on the order chosen. The order adopted here is to initially maximize the performance index with respect to $\mathbf{x}(0)$ and $\mathbf{w}$, and then perform a min-max with respect to $\hat{\mathbf{x}}$ and $\mathbf{y}$. After the first stage of optimization, the optimal values of $\mathbf{x}(0)$ and $\mathbf{w}$ are plugged into the performance index, and then the min-max operation with respect to $\hat{\mathbf{x}}$ and $\mathbf{y}$ is performed.

The order chosen is justified as follows. Amongst the three adversaries $(\mathbf{y}, \mathbf{w}$ and $\mathbf{x}(0))$, the estimator has complete information of only the measurement $\mathbf{y}$. Consequently, it operates directly on the measurement. The estimator lacks any information of $\mathbf{w}$ and $\mathbf{x}(0)$. Hence, it should be prepared for the worst possible $\mathbf{w}$ and $\mathbf{x}(0)$. Allowing these two players, $\mathbf{w}$ and $\mathbf{x}(0)$, to go first would help determine their best strategies. The subsequent min-max operation between the players $\hat{\mathbf{x}}$ and $\mathbf{y}$ results in an estimator which is a function of the measurements.

3. Solution to the Estimation Game Problem

a. Maximization with x(0) and w

Consider first, the maximization of J with respect to $\mathbf{w}$ and $\mathbf{x}(0)$, for a fixed $\hat{\mathbf{x}}$ and $\mathbf{y}$. Let

$$J = \max_{\mathbf{w}} \max_{\mathbf{x}(0)} J \tag{9}$$

The standard variational procedure [8] is formally applied. A Lagrange multiplier is introduced to adjoin the constraint(1) to the performance index. The resulting Hamiltonian is

$$H = \frac{1}{2} \| \mathbf{x} - \hat{\mathbf{x}} \|^2_{\bar{Q}} - \frac{1}{2\theta}(\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{y} - C\mathbf{x} \|^2_{\bar{V}}) + \frac{\lambda^T}{\theta}(A\mathbf{x} + B\mathbf{w}) \tag{10}$$

where $\frac{\lambda^T}{\theta}$ is a Lagrange multiplier. Taking the first variation, the first order necessary conditions for a maximum are

$$\mathbf{x}(0) = \hat{\mathbf{x}}_0 + P_0\lambda(0), \quad \lambda(T) = 0$$
$$\mathbf{w} = WB^T\lambda$$
$$\dot{\lambda} = -A^T\lambda - \theta\bar{Q}(\mathbf{x} - \hat{\mathbf{x}}) - C^T\bar{V}(y - Cx) \tag{11}$$

These first order necessary conditions result in a two-point boundary value problem

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & BWB^T \\ C^T\bar{V}C - \theta\bar{Q} & -A^T \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ \theta\bar{Q}\hat{\mathbf{x}} - C^T\bar{V}\mathbf{y} \end{pmatrix} \tag{12}$$

with boundary conditions

$$\mathbf{x}(0) = \hat{\mathbf{x}}_0 + P_0\lambda(0), \quad \lambda(T) = 0 \tag{13}$$

Since the two-point boundary value problem is linear, the solution is assumed to be of the form

$$\mathbf{x}^* = \mathbf{x}_p + P\lambda^* \tag{14}$$

where $\mathbf{x}_p$ and P are undetermined variables of appropriate dimension. $\mathbf{x}^*$ and λ^* represent optimal values of $\mathbf{x}$ and λ, respectively, for any fixed admissible functions of $\hat{\mathbf{x}}$ and $\mathbf{y}$. The optimal values for $\mathbf{w}$ and $\mathbf{x}(0)$ are

$$\mathbf{w}^* = WB^T\lambda^*, \quad \mathbf{x}(0)^* = \hat{\mathbf{x}}_0 + P_0\lambda(0)^* \tag{15}$$

Differentiating (14) and substituting for $\dot{\mathbf{x}}^*$ and $\dot{\lambda}^*$ from (12) results in

$$\dot{\mathbf{x}}_p - A\mathbf{x}_p - PC^T\bar{V}(\mathbf{y} - C\mathbf{x}_p) + \theta P\bar{Q}(\mathbf{x}_p - \hat{\mathbf{x}}) = [AP + PA^T + BWB^T - P(C^T\bar{V}C - \theta\bar{Q})P - \dot{P}]\lambda^* \tag{16}$$

For equation (16) to hold true for arbitrary λ^*, the left hand side and right hand are set identically to zero, resulting in

$$\dot{\mathbf{x}}_p = A\mathbf{x}_p + PC^T\bar{V}(\mathbf{y} - C\mathbf{x}_p) - \theta P\bar{Q}(\mathbf{x}_p - \hat{\mathbf{x}}); \quad \mathbf{x}_p(0) = \hat{\mathbf{x}}_0 \tag{17}$$

$$\dot{P} = AP + PA^T + BWB^T - P(C^T\bar{V}C - \theta\bar{Q})P; \quad P(0) = P_0 \tag{18}$$

where (18) is the well-known Riccati differential equation. Note that $P(t)$ is symmetric.

Claim 1 If the solution $P(t)$ to the Riccati differential equation (18) exists $\forall t \in [0, T]$, then $P(t) > 0 \ \forall t \in [0, T]$.

Proof: See Rhee and Wonham [9, 11].

Claim 1 is used extensively in the proofs to follow. In section 7, the existence of $P(t) \ \forall t \in [0, T]$ is shown to be both a necessary and sufficient condition for optimality.

b. Min-Max w.r.t. $\hat{\mathbf{x}}$ and y

The optimal strategies

$$\mathbf{w}^* = WB^T\lambda^*, \quad \mathbf{x}(0)^* = \hat{\mathbf{x}}_0 + P_0\lambda(0)^*$$

from (15) are substituted into the performance index, and adding the identically zero term

$$\frac{1}{2\theta} \| \lambda^*(0) \|^2_{P_0} - \frac{1}{2\theta} \| \lambda^*(T) \|^2_{P(T)} + \frac{1}{2\theta}\int_0^T d/dt(\| \lambda^*(t) \|^2_{P(t)})dt = 0$$

results in the min-max problem

$$\min_{\hat{\mathbf{x}}} \max_{\mathbf{y}} J = \frac{1}{2}\int_0^T (\| \mathbf{x}_p - \hat{\mathbf{x}} \|^2_{\bar{Q}} - \frac{1}{\theta} \| (\mathbf{y} - C\mathbf{x}_p) \|^2_{\bar{V}})dt \tag{19}$$

subject to the dynamic constraints

$$\dot{\mathbf{x}}_p = A\mathbf{x}_p + PC^T\bar{V}(\mathbf{y} - C\mathbf{x}_p) - \theta P\bar{Q}(\mathbf{x}_p - \hat{\mathbf{x}}); \mathbf{x}_p(0) = \hat{\mathbf{x}}_0 \tag{20}$$

$$\dot{P} = AP + PA^T + BWB^T - P(C^T\bar{V}C - \theta\bar{Q})P; P(0) = P_0 \tag{21}$$

By executing a change of variables, the min-max problem is recast into one with a more insightful structure. The change of variables

$$\mathbf{r} = \mathbf{x}_p - \hat{\mathbf{x}},$$
$$\mathbf{q} = \mathbf{y} - C\mathbf{x}_p \tag{22}$$

gives

$$\min_{\mathbf{r}} \max_{\mathbf{q}} J = \frac{1}{2}\int_0^T (\| \mathbf{r} \|^2_{\bar{Q}} - \frac{1}{\theta} \| \mathbf{q} \|^2_{\bar{V}})dt \tag{23}$$

subject to

$$\dot{\mathbf{x}}_p = A\mathbf{x}_p + PC^T\bar{V}\mathbf{q} - \theta P\bar{Q}\mathbf{r}; \quad \mathbf{x}_p(0) = \hat{\mathbf{x}}_0 \tag{24}$$

$$\dot{P} = AP + PA^T + BWB^T - P(C^T\bar{V}C - \theta\bar{Q})P; \quad P(0) = P_0 \tag{25}$$

The new form (23) has a separable performance index [8]. The two independent players, $\mathbf{r}$ and $\mathbf{q}$, affect the variable $\mathbf{x}_p$. However, $\mathbf{x}_p$ does not appear in the performance index. Therefore, the optimal strategies of $\mathbf{r}$ and $\mathbf{q}$ are readily seen to be

$$\mathbf{r}^* = 0; \quad \mathbf{q}^* = 0 \tag{26}$$

From (22) and (26)

$$\hat{\mathbf{x}}^* = \mathbf{x}_p; \quad \mathbf{y}^* = C\mathbf{x}_p \tag{27}$$

The value of the game is the value of the cost function, when all the players use their optimal strategies. When the optimal strategies $\hat{x}^*, y^*, w^*, x(0)^*$ given by

$$\begin{aligned}\hat{x}^* = x_p; \; y^* &= Cx_p \\ w^* = WB^T\lambda^*, \quad x(0)^* &= \hat{x}_0 + P_0\lambda(0)^* \\ x^* &= x_p + P\lambda^* \end{aligned} \tag{28}$$

are substituted into the cost function,

$$J(\hat{x}^*, y^*, w^*, x(0)^*) = 0 \tag{29}$$

giving a zero value game.

c. The Saddle-point inequality

The strategies $\hat{x}^*, y^*, w^*, x(0)^*$ have been assumed to be optimal so far, based on satisfying the necessary conditions for optimality. If the strategies satisfy a saddle-point inequality, they represent optimal strategies. Satisfying a saddle-point also proves that the strategies are optimal irrespective of the order of optimization [8].

Claim 2 If $P(t)$ exists $\forall t \in [0,T]$, the optimal strategies $\hat{x}^*, y^*, w^*, x(0)^*$ satisfy a saddle-point inequality,

$$\begin{aligned} J(\hat{x}^*, y, w, x(0)) \le & J(\hat{x}^*, y^*, w^*, x(0)^*) \\ \le & J(\hat{x}, y^*, w^*, x(0)^*) \end{aligned} \tag{30}$$

where $\hat{x}^*, y^*, w^*, x(0)^*$ denote optimal strategies and $\hat{x}, y, w, x(0)$ denote any admissible strategies.

Proof: From (29),

$$J(\hat{x}^*, y^*, w^*, x(0)^*) = 0$$

Consider the right inequality

$$J(\hat{x}^*, y^*, w^*, x(0)^*) \le J(\hat{x}, y^*, w^*, x(0)^*)$$

where

$$\begin{aligned} J(\hat{x}, y^*, w^*, x(0)^*) = & -\frac{1}{2\theta} \| x(0)^* - \hat{x}_0 \|^2_{P_0^{-1}} \\ + \frac{1}{2}\int_0^T & \| x^* - \hat{x} \|^2_Q - \frac{1}{\theta}(\| w^* \|^2_{W^{-1}} + \| y^* - Cx^* \|^2_V) dt \end{aligned} \tag{31}$$

By adding the identically zero term

$$\begin{aligned} \frac{1}{2\theta} \| x(0)^* - \hat{x}_0 \|^2_{P_0^{-1}} - \frac{1}{2\theta} \| x(T)^* - \hat{x}(T) \|^2_{P^{-1}(T)} \\ + \frac{1}{2\theta}\int_0^T \frac{d}{dt}(\| x^*(t) - \hat{x}(t) \|^2_{P^{-1}(t)}) dt \end{aligned}$$

where $x(t)^*$ and $\lambda^*(t)$ represent the optimal trajectories of $x(t)$ and $\lambda(t)$, to (31), the expression reduces to

$$\begin{aligned} J(\hat{x}, y^*, w^*, x(0)^*) = \frac{1}{2\theta}\int_0^T & (2 \| x^* - \hat{x} \|^2_Q + \\ \| P\lambda^* \|^2_Q - \| & (x^* - \hat{x}) - P\lambda^* \|^2_Q) dt \end{aligned} \tag{32}$$

Since $\| x^* - \hat{x} \|^2_Q + \| P\lambda^* \|^2_Q \ge \| (x^* - \hat{x}) - P\lambda^* \|^2_Q$, the integrand is always ≥ 0 and therefore

$$J(\hat{x}, y^*, w^*, x(0)^*) \ge 0 \tag{33}$$

which proves the right inequality.

Adopting a similar procedure as above, it is easily shown that

$$\begin{aligned} J(\hat{x}^*, y, w, x(0)) = & -\frac{1}{2\theta} \| x(T) - \hat{x}^*(T) \|^2_{P^{-1}(T)} \\ - \frac{1}{2\theta}\int_0^T & (\| WB^T P^{-1}(x - \hat{x}^*) - w \|^2_{W^{-1}} \\ & + \| y - C\hat{x}^* \|^2_V) dt \end{aligned} \tag{34}$$

By assumption $P(t)$ exists $\forall t \in [0,T]$. Further, from claim 1, $P^{-1}(t)$ exists $\forall t \in [0,T]$ and $P^{-1}(T) > 0$. Therefore

$$J(\hat{x}^*, y, w, x(0)) \le 0 \tag{35}$$

which proves the left inequality. From (29), (33) and (35), the claim is proved. QED.

d. Extension to a Time-invariant System

In a straight forward way, the results of the previous sections can be extended to a linear time-invariant system on an infinite horizon. The following assumptions are made

1. A, B, C, D are constant matrices and $T \to \infty$
2. (A, B) is controllable and (C, A) is detectable
3. Q, W and V are identity matrices

Assumption 1 characterizes a time-invariant system. Assumption 2 is essential to obtain a stable filter and establish optimality. Assumption 3 helps establish the H_∞ norm bound without complications.

The optimal steady state strategies are

$$\begin{aligned}\hat{x}^* = x_p; \; y^* &= Cx_p \\ w^* = WB^T\lambda^*, \quad x(0)^* &= \hat{x}_0 + P_0\lambda(0)^* \\ x^* &= x_p + P_{ss}\lambda^* \end{aligned} \tag{36}$$

where P_{ss} is the steady state solution to the Riccati differential equation (18) for a given initial condition P_0. It can be readily seen from Claim 1 that the steady solution, if it exists, is positive definite. If P_0 does not coincide with any of the positive definite solutions to the algebraic Riccati equation [11], it appears that the solution to the Riccati differential equation, if it converges, converges to the smallest solution to the algebraic Riccati equation. Furthermore, if P_0 is less than the minimal positive definite solution, then the existence of a steady state solution is guaranteed [11]. P_0, in a sense, reflects the strength of the adversary $x(0)$. As P_0 increases, the adversary $x(0)$ has a greater advantage. There may exist a critical value for P_0 above which the saddle point solution for the infinite time game will not exist.

Setting $\dot{P} = 0$ (steady state), (18) reduces to an algebraic equation

$$AP + PA^T + BB^T - P(C^T D^{-T} D^{-1} C - \theta L^T L)P = 0 \tag{37}$$

Assuming $P_{ss} = P_0$ is a positive definite solution to (37), equality (32) reduces to

$$\begin{aligned} J(\hat{x}, y^*, w^*, x(0)^*) = \\ \frac{1}{2\theta}\int_0^\infty (2 \| x^* - \hat{x} \|^2_I + \| P_{ss}\lambda^* \|^2_I - \| (x^* - \hat{x}) \\ - P_{ss}\lambda^* \|^2_I) dt \end{aligned}$$

and equality (34) reduces to

$$\begin{aligned} J(\hat{x}^*, y, w, x(0)) = -\frac{1}{2\theta} \| x(\infty) - \hat{x}^*(\infty) \|^2_{P_{ss}^{-1}} \\ - \frac{1}{2\theta}\int_0^\infty (\| B^T P_{ss}^{-1}(x - \hat{x}^*) - w \|^2_I + \| y - C\hat{x}^* \|^2_{D^{-T}D^{-1}}) dt \end{aligned} \tag{38}$$

and the saddle-point inequality

$$\begin{aligned} J(\hat{x}^*, y, w, x(0)) \le & J(\hat{x}^*, y^*, w^*, x(0)^*) \\ \le & J(\hat{x}, y^*, w^*, x(0)^*) \end{aligned}$$

still holds.

Note that $P_{ss} > 0$ is a sufficient condition for the saddle-point inequality to hold.

4. The Optimal Estimator

a. Time-varying System

The optimal estimate $\hat{x}^*$, from (24) and (27), is governed by

$$\dot{\hat{x}}^* = A\hat{x}^* + PC^T\bar{V}(y - C\hat{x}^*); \;\; \hat{x}^*(0) = \hat{x}_0 \tag{39}$$

with

$$\begin{aligned} \dot{P} = AP + PA^T + BWB^T - P(C^T\bar{V}C - \theta\bar{Q})P \\ P(0) = P_0 \end{aligned} \tag{40}$$

The optimal estimator (39) is seen to be both linear as well as unbiased. Its structure is similar to that of a Kalman filter; the distinguishing feature, however, is the dependence of the estimate on $\bar{Q}$, the weighting on the estimation error in the performace measure. $\bar{Q}$ affects the dynamics of P (40); P in turn affects $\hat{x}^*$. $\bar{Q} = L^T Q L$ and the optimal estimate of $z = Lx$ is dependent on L.

b. Time-invariant System

Now consider the time-invariant case (A, B, C, D are constant matrices), and let $T \to \infty$. Let W, V and Q be identity matrices. (A, B) is controllable and (C, A) is detectable. The optimal steady state estimator is given by

$$\dot{\hat{x}}^* = A\hat{x}^* + P_{ss}C^T D^{-T} D^{-1}(y - C\hat{x}^*); \;\; \hat{x}^*(0) = \hat{x}_0 \tag{41}$$

where $P_{ss} > 0$ is a positive definite solution to the algebraic Riccati equation

$$AP + PA^T + BB^T - P(C^T D^{-T} D^{-1} C - \theta L^T L)P = 0 \tag{42}$$

Once again, note that the optimal estimate is dependent on L.

Claim 3 The matrix $(A - P_{ss}C^T D^{-T} D^{-1} C)$ is stable.

Proof: The algebraic Riccati equation (37) can be written as

$$\begin{aligned} (A - P_{ss}C^T D^{-T} D^{-1} C)P_{ss} + P(A - P_{ss}C^T D^{-T} D^{-1} C)^T \\ = -[BB^T + P_{ss}C^T D^{-T} D^{-1} C P_{ss} + \theta P_{ss} L^T L P_{ss}] \end{aligned} \tag{43}$$

which is a Lyapunov equation. By assumption, (A, B) is controllable. Note that $[BB^T + P_{ss}C^T D^{-T} D^{-1} C P_{ss} + \theta P_{ss} L^T L P_{ss}]$ is a symmetric, positive semi-definite matrix. From lemma 4.1 in Wonham [9], $(A, [BB^T + P_{ss}C^T D^{-T} D^{-1} C P_{ss} + \theta P_{ss} L^T L P_{ss}]^{\frac{1}{2}})$ is controllable. Once again applying lemma 4.1 [9], $(A - P_{ss}C^T D^{-T} D^{-1} C, [BB^T + P_{ss}C^T D^{-T} D^{-1} C P_{ss} + \theta P_{ss} L^T L P_{ss}]^{\frac{1}{2}})$ is controllable. Let

$$\begin{aligned} \bar{A} &= (A - P_{ss}C^T D^{-T} D^{-1} C) \\ H &= [BB^T + P_{ss}C^T D^{-T} D^{-1} C P_{ss} + \theta P_{ss} L^T L P_{ss}]^{\frac{1}{2}} \end{aligned}$$

(43) reduces to

$$\bar{A}P_{ss} + P_{ss}\bar{A}^T = -HH^T \tag{44}$$

From Chen [13], corollary 8.20, the eigenvalues of $\bar{A}$ have negative real parts, if and only if for any given positive semi-definite matrix $N = (HH^T)$ with the property $(\bar{A}, N)$ controllable, the matrix

$$\bar{A}P + P\bar{A}^T = -N$$

has a unique positive definite solution for P.

Since $P_{ss} > 0$, $\bar{A}$ is a stable matrix.

QED.

The dependence on L distinguishes the H_∞ estimator from the H_2 estimator. In H_2 estimation, the optimal estimator produces the best estimate of all the states, independent of L. In H_∞ estimation, the optimal estimator produces the best estimate of that particular combination of states whose estimate is sought. Further more, in H_2 estimation the stabilizing, positive definite solution to the algebraic Riccati equation is unique, while in H_∞ estimation more than one stabilizing, positive definite solution could exist. The minimal positive definite solution should be used to insure the smallest estimator bandwidth.

In the limiting case, where the parameter $\theta \to 0$, the optimal estimators given by (39) and (41) reduce to a Kalman filter and a steady state Kalman filter respectively. And, in a stochastic sense, the equation (40) reduces to the Riccati equation governing the propogation of the covariance of the estimation error and P_{ss} denotes the steady state covariance matrix for the time-invariant case. In particular, note that as $\theta \to 0$, the optimal estimate is independent of the weighting $\bar{Q}$ on the error.

The optimal strategy of the measurement noise

$$\begin{aligned} \mathbf{v}^* &= D^{-1}(\mathbf{y}^* - C\mathbf{x}_p) \\ \Rightarrow \mathbf{v}^* &= D^{-1}(C\mathbf{x}_p - C\mathbf{x}_p) = 0 \end{aligned} \tag{45}$$

The restriction that the optimal estimate belong to a class of functions, which are functions of the measurement alone, is not explicitly introduced into the optimization procedure. The optimal estimator is a function of the measurement as a consequence of the order of optimization adopted. This point helps explain the optimal strategy of $\mathbf{v}$. Note that $\mathbf{v}$ affects only the quadratic term $\int_0^T \frac{-1}{2\theta}(\| \mathbf{v} \|^2_{V^{-1}})dt$ in the performance index (8) . $\mathbf{v}$ is not cognizant of the fact that $\hat{\mathbf{z}}$'s strategy depends on the measurement. Therefore, to maximize the index, the value of $\mathbf{v}$ is straightaway seen to be zero.

5. Finite Time Interval Smoothing

The optimal smoother, for a finite time interval, is now derived. The smoother $\hat{\mathbf{x}}_s(s)$, $0 \le s \le T$ uses all the information available from the measurements over the interval $[0, T]$. The optimal smoother accounts for the worst possible process noise $\mathbf{w}$ and uncertainty in the initial state.

In terms of the game formulation (8), the problem reduces to a one-sided optimization

$$\max_{\mathbf{w},\mathbf{x}(0)} J = -\frac{1}{2\theta} \| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \frac{1}{2}\int_0^T \| \mathbf{x} - \hat{\mathbf{x}}_s \|^2_{\bar{Q}} - \frac{1}{\theta}(\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{y} - C\mathbf{x} \|^2_{\bar{V}})dt \tag{46}$$

subject to

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{w} \tag{47}$$

Note that the estimate $\hat{\mathbf{x}}$ is replaced by the smoothed estimate $\hat{\mathbf{x}}_s$ and after the optimization is performed, $\mathbf{x}^*$ is replaced with $\hat{\mathbf{x}}_s$. $\mathbf{x}^*$ is the state trajectory when $\mathbf{w}$ and $\mathbf{x}(0)$ play their optimal strategies. Given all the available information $\{Y_T = (\mathbf{y}(s) : 0 \le s \le T)\}$, the smoothed estimate follows that state trajectory which corresponds to the worst process noise and initial state, i. e. $\hat{\mathbf{x}}_s = \mathbf{x}_{worst}$.

As before (section 3), formally applying the variational procedure, the first order necessary conditions along with $\hat{\mathbf{x}}_s = \mathbf{x}^*$ result in a two-point boundary value problem

$$\begin{pmatrix} \dot{\hat{\mathbf{x}}}_s \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & BWB^T \\ C^T\bar{V}C & -A^T \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_s \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ -C^T\bar{V}\mathbf{y} \end{pmatrix} \tag{48}$$

with boundary conditions

$$\hat{\mathbf{x}}_s(0) = \hat{\mathbf{x}}_0 + P_0\lambda(0), \quad \lambda(T) = 0 \tag{49}$$

Note that the smoothed estimate is independent of the weighting $\bar{Q}$ on the estimation error in the attenuation function. Since the smoothed estimate is independent of $\bar{Q}$, it is independent of L, the linear combination of states whose estimate is sought. This goes to prove the claim in [7]. Further, this feature makes the H_∞ smoother identical to the H_2 smoother.

6. The Optimal Estimate and the Attenuation Function

The optimal estimate is substituted into the attenuation function and the resulting expression examined.

a. Time-varying System

Claim 4 Given that $P(t)$ exists $\forall t \in [0, T]$, the optimal estimate satisfies the inequality

$$J_{of}(\hat{\mathbf{z}}^*, \mathbf{w}, \mathbf{v}, \mathbf{x}(0)) < \frac{1}{\theta} \tag{50}$$

$\forall$ admissible $(\mathbf{w}, \mathbf{v}, \mathbf{x}(0)) \neq (\mathbf{w}^*, \mathbf{v}^*, \mathbf{x}(0)^*)$.

Remark: Note that the values of the optimal strategies on the saddle $(\hat{\mathbf{x}}^*, \mathbf{w}^*, \mathbf{v}^*, \mathbf{x}(0)^*) = (0, 0, 0, \hat{\mathbf{x}}_0)$ and therefore, $((\mathbf{z}^* - \hat{\mathbf{z}}^*), (\mathbf{x}(0)^* - \hat{\mathbf{x}}_0), \mathbf{w}^*, \mathbf{v}^*) = 0$.

Proof: From (35)

$$J(\hat{\mathbf{x}}^*, \mathbf{y}, \mathbf{w}, \mathbf{x}(0)) < 0$$

In terms of the original cost-function

$$J(\hat{\mathbf{z}}^*, \mathbf{y}, \mathbf{w}, \mathbf{x}(0)) < 0$$

and rearranging gives

$$J_{of} = \frac{\int_0^T \| \mathbf{z} - \hat{\mathbf{z}}^* \|^2_Q \, dt}{\| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \int_0^T \| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{v} \|^2_{V^{-1}} \, dt} < \frac{1}{\theta} \tag{51}$$

$$J_{of}(\hat{\mathbf{z}}^*, \mathbf{w}, \mathbf{v}, \mathbf{x}(0)) < \frac{1}{\theta}$$

QED.

b. Time-invariant System

Consider the system to be time-invariant (A, B, C, D are constant matrices), and let $T \to \infty$. Q, W and V are identity matrices. (A, B) is controllable and (C, A) is detectable.

Claim 5 Given that $P_{ss} > 0$ is a positive definite solution to the algebraic Riccati equation (37), the optimal estimate satisfies the inequality

$$\| T_{ed} \|_\infty < \frac{1}{\sqrt{\theta}}$$

$\forall$ $\mathbf{w}, \mathbf{v} \in L_2$, where $\| T_{ed} \|_\infty$ is the H_∞ norm of the transfer function matrix from the external signals to the estimation error.

Proof: (??) can be rewritten as

$$-\frac{1}{2\theta} \| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \frac{1}{2}\int_0^\infty \| \mathbf{z} - \hat{\mathbf{z}}^* \|^2_I - \frac{1}{\theta}(\| \mathbf{w} \|^2_I + \| \mathbf{v} \|^2_I)dt < 0 \tag{52}$$

Rearranging gives

$$J_{of} = \frac{\int_0^\infty \| \mathbf{z} - \hat{\mathbf{z}}^* \|^2_I \, dt}{\| \mathbf{x}(0) - \hat{\mathbf{x}}_0 \|^2_{P_0^{-1}} + \int_0^\infty \| \mathbf{w} \|^2_I + \| \mathbf{v} \|^2_I \, dt} < \frac{1}{\theta} \tag{53}$$

and consequently

$$\| T_{ed} \|_\infty < \frac{1}{\sqrt{\theta}} \tag{54}$$

QED.

Remark: Since $\mathbf{w}, \mathbf{v} \in L_2$, all positive definite P_{ss} satisfy (54).

7. Optimality Condition

The existence of $P(t)$ $\forall t \in [0, T]$ has been assumed in all the proofs so far. In this section this is shown to be both a necessary and sufficient condition for optimality.

Claim 6 The existence of $P(t)$ $\forall t \in [0, T]$ is both a necessary and sufficient condition for optimality.

Proof: For $\hat{\mathbf{x}}^*$ to be optimal,

$$\Delta J = J(\hat{\mathbf{x}}^*, \mathbf{y}, \mathbf{w}, \mathbf{x}(0)) - J(\hat{\mathbf{x}}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{x}(0)^*) \le 0 \quad \forall \mathbf{y}, \mathbf{w} \in L_2[0, T] \ \ and \ \ \mathbf{x}(0) \in R^n.$$

(Sufficiency): Shown in claim 2.

(Necessity): Assume that $P(t)$ becomes unbounded at time t_e, $0 \le t_e \le T$. $P(t) \to \infty$ as $t \to t_e$ from below. Let ϵ be a small number. The variation

$$\Delta J = \lim_{\epsilon \to 0}\Big(\frac{-1}{2\theta} \| \Delta\mathbf{e}^*(t_e - \epsilon) \|^2_{P^{-1}(t_e - \epsilon)} - \frac{1}{2\theta}\int_0^{t_e - \epsilon} \| WB^TP^{-1}(\Delta\mathbf{e}^*) - \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{y} - C\hat{\mathbf{x}}^* \|^2_{\bar{V}})dt + \frac{1}{2}\int_{t_e - \epsilon}^T \| \Delta e^* \|^2_{\bar{Q}} - \frac{1}{\theta}(\| \mathbf{w} \|^2_{W^{-1}} + \| \mathbf{y} - C\mathbf{x} \|^2_{\bar{V}})dt \tag{55}$$

where $\Delta\mathbf{e}^* = \mathbf{x} - \hat{\mathbf{x}}^*$ represents the error in the optimal estimate. The existence of $P(t)$ is not assumed for the interval $[t_e - \epsilon, T]$ and consequently the optimal estimator given by (39) and (40) does not exist in this interval. In the interval $[t_e - \epsilon, T]$, $\hat{\mathbf{x}}^*$ represents the optimal strategy of $\hat{\mathbf{x}}$, no longer governed by the dynamics (39).

Now consider the following strategies for $\mathbf{y}$ and $\mathbf{w}$,

$$\begin{aligned} \mathbf{w} = WB^TP^{-1}\Delta\mathbf{e}^*, \mathbf{y} = C\hat{\mathbf{x}}^* \quad &\forall t \in [0, t_e - \epsilon] \\ \mathbf{w} = 0, \mathbf{y} = C\mathbf{x} \quad &\forall t \in [t_e - \epsilon, T] \end{aligned} \tag{56}$$

Then

$$\Delta J = \lim_{\epsilon\to 0}(\frac{-1}{2\theta} \parallel \Delta e^*(t_e-\epsilon) \parallel^2_{P^{-1}(t_e-\epsilon)} + \frac{1}{2}\int_{t_e-\epsilon}^{T} \parallel \Delta e^* \parallel^2_Q dt) \quad (57)$$

As $\epsilon \to 0$, $P^{-1}(t_e-\epsilon)$ tends to a singular matrix. For the interval, $[0, t_e-\epsilon]$, Δe^* is governed by

$$\begin{aligned} \Delta\dot{e}^* &= A\Delta e^* + B(WB^TP^{-1}\Delta e^*) \\ &= (A+BWB^TP^{-1})\Delta e^* \end{aligned} \quad (58)$$

The solution to the linear dynamic equation (58) can be expressed in terms of a state-transition matrix,

$$\Delta e^*(t_e-\epsilon) = \Phi((t_e-\epsilon),0)\Delta e^*(0) \quad (59)$$

where $\Phi(.,.)$ is the state-transition matrix of $(A+BWB^TP^{-1})$ and $\Delta e^*(0) = x(0) - \hat{x}_0$.

Since $x(0)$ is arbitrary, $x(0)$ can be chosen such that $\lim_{\epsilon\to 0}\Delta e^*(t_e-\epsilon)$ lies in the null space of $\lim_{\epsilon\to 0} P^{-1}(t_e-\epsilon)$. Note that the latter is a singular matrix. Then

$$\lim_{\epsilon\to 0}(\frac{-1}{2\theta} \parallel \Delta e^*(t_e-\epsilon) \parallel^2_{P^{-1}(t_e-\epsilon)} = 0 \quad (60)$$

and

$$\Delta J = \lim_{\epsilon\to 0}\frac{1}{2}\int_{t_e-\epsilon}^{T} \parallel \Delta e^* \parallel^2_Q dt > 0 \quad (61)$$

Therefore, (61) is a contradiction that $\hat{x}^*$ is optimal. Hence, $P(t)$ must exist $\forall t \in [0,T]$.

For the time-invariant system, existence of a positive definite solution P_{ss} to the algebraic Riccati equation (37) is both a necessary and sufficient condition for a stable filter. (see Claim 3). $P_{ss} > 0$ is a sufficient condition for optimality.

8.Conclusions

A robust estimator and smoother, satisfying an upper bound on a disturbance attenuation function have been derived from a quadratic game formulation. For a linear time- invariant system, the upper bound on the disturbance attenuation function is equivalent to a bound on the H_∞ norm.The optimal estimator, obtained from a general class of nonlinear functions of the measurements, is linear and unbiased. Both necessary and sufficient conditions for optimality are presented. For the time-varying case, the existence of a solution to the Riccati differential equation, over the time interval, is a necessary and sufficient condition for the existence of the optimal filter. For the time-invariant case, the existence of a positive definite solution to the algebraic Riccati equation is a necessary and sufficient condition for the existence of a stable and optimal filter.

The optimal estimator is dependant on the weighting on the estimation error in the performance criterion, while the optimal smoother is independent of the weighting on the estimation error. This shows that the optimal estimate is dependant on the linear combination (L) of the states whose estimate is sought, while the optimal smoother gives the best smoothed estimate of every state independent of L. If the disturbances are white noise processes, the smoother obtained is identical to the white-noise smoother. In the limiting case when $\theta \to 0$ and the disturbances are white noise processes, the optimal estimator reduces to a Kalman filter.

Appendix

The following explaination is an attempt to link the frequency domain concept of H_∞ synthesis to the equivalent LQ game concept in the time domain.

Consider the causal, linear time-invariant system,

$$\dot{x} = Ax + Bw \quad (62)$$

with a linear measurement

$$y = Cx + Dv \quad (63)$$

where $x(t) \in R^n$, $w(t) \in R^m$, $v(t) \in R^p$. $w(.), v(.) \in L_2[0,\infty)$. $A \in R^{n\times n}$, $B \in R^{n\times n}$, $C \in R^{p\times n}$, $D \in R^{p\times p}$. (A,B) is stabilizable and (C,A) is detectable. D is assumed to be invertible. The optimal estimate of the vector $z = Lx$ is sought, where $z(t) \in R^q$ and $L \in R^{q\times n}$. Let $d = [w^T, v^T]^T$ represent the disturbances, $e = z - \hat{z}$ the estimation error. If F is the transfer function matrix from the disturbances d to the estimate error e, then

$$\bar{e} = F\bar{d} \quad (64)$$

where $\bar{e}, \bar{d}$ are in the Laplace domain.

The energy of the external signals, being in $L_2[0,\infty)$, is bounded. The energy of the signal e evaluated in the time domain

$$\parallel e \parallel_2 = [\int_0^\infty e^*(t)e(t)dt]^{\frac{1}{2}} \quad (65)$$

and in the frequency domain

$$\parallel \bar{e} \parallel_2 = [\int_{-\infty}^{\infty} \bar{e}^*(j\omega)\bar{e}(j\omega)dt]^{\frac{1}{2}} \quad (66)$$

where $*$ denotes the complex-conjugate transpose, and

$$\parallel e \parallel_2 = \parallel \bar{e} \parallel_2 < \infty \quad (67)$$

A fundamental fact is that the $L_2[0,\infty)$ gain of a causal, linear time-invariant system equals the H_∞ norm of its transfer function, i. e.

$$\begin{aligned} \parallel F \parallel_\infty &= \sup_{\bar{d}}[\parallel F\bar{d} \parallel_2 : \parallel \bar{d} \parallel_2 = 1] \\ &= \sup_{\bar{d}}[\parallel \bar{e} \parallel_2 : \parallel \bar{d} \parallel_2 = 1] \\ &= \sup_{d}[\parallel e \parallel_2 : \parallel d \parallel_2 = 1] \end{aligned} \quad (68)$$

Therefore,

$$\min_{\hat{z}} \parallel F \parallel_\infty = \min_{\hat{z}} \sup_{d}[\parallel e \parallel_2 : \parallel d \parallel_2 = 1] \quad (69)$$

Note that (69) links the frequency domain to the time domain and helps extend the interpretation of H_∞ norm minimization to causal, linear time-varying systems.

For a causal,linear time-varying system the optimization problem is set up as

$$\min_{\hat{z}} J = \min_{\hat{z}} \sup_{d} \parallel e \parallel_2 \quad (70)$$

subject to

$$\parallel v \parallel_2 + \parallel w \parallel_2 \leq 1 \quad (71)$$

Since $\hat{z}^*$, the optimal estimate, should anticipate the worst w, v, the $L_2[0,\infty)$ constraint is incorporated into the performance measure with the aid of a scalar Lagrange multiplier $\frac{1}{\theta}$ and the problem is recast into a min-max form,

$$\min_{\hat{z}} \max_{(w,v)} J = \parallel e \parallel_2 - \frac{1}{\theta}(\parallel v \parallel_2 + \parallel w \parallel_2) \quad (72)$$

As shown in this paper, a saddle-point solution to the above problem, ensures that

$$J(e^*, v, w) \leq J(e^*, v^*, w^*) \leq J(e, v^*, w^*) \quad (73)$$

and the left hand side inequality ensures that

$$\parallel e \parallel_2 < \frac{1}{\theta} \quad (74)$$

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