EE6323 Feb-May, 2021

Class #13 (May 11, 2021)

Note Title 27-03-2020

Channel Estimation → OFDM

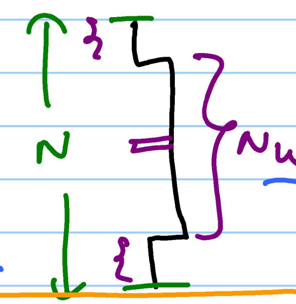
OFDM CE →
- Scalar model (no correlation is exploited)
- vector model (1-D) (freq. correlation is exploited)
- Matrix model (2-D) (both time & freq. corr. are exploited)

CIR ⟷ CFR

$h_l[k],\ l = 0 \ldots L-1$ ($N_{cp}-1$)

$h[k,m] \leftarrow h(\tau) \rightleftharpoons H(f)$

$H[k,n] \leftarrow h(\tau;t) \rightleftharpoons H(f;t)$ → k+1, k+2.



Scalar model → $Y[k,n] = H[k,n] \cdot I[k,n] + V[k,n]$ $\forall n = 1 \ldots N_u$

$H[k,n-1]$, $H[k,n+1]$, $I_p[k,n]$

OFDM-CE

- semi-blind, decision-directed
- Supervised (pilot-Assisted)
- Blind
  - CE ← cumulants
- sub-spaced multipath model
  - $h(\tau)$ (non-sample spaced)
- sample-spaced (T_s)
  - multi-tap model
  - $h[m] \{h_\ell\}$, mT_s

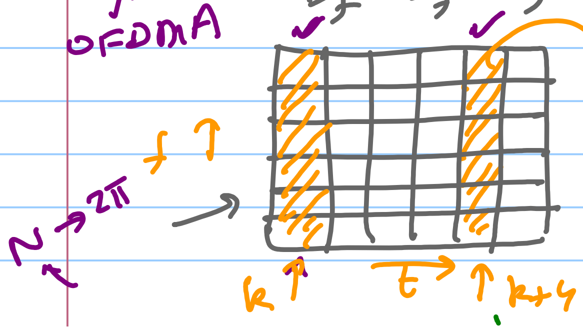
Pilot subcarriers

OFDM

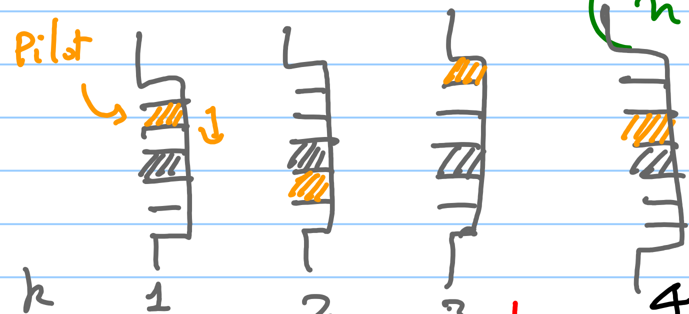
OFDMA

$f_c - f_c' = \Delta f$

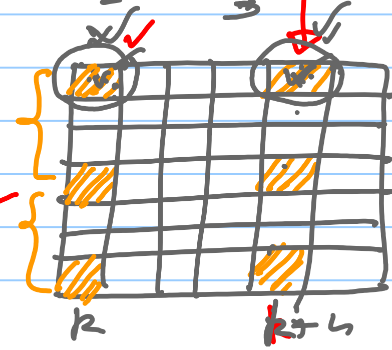
$\Delta f - \hat{\Delta f} = \tilde{\Delta f}$

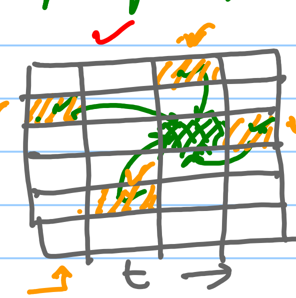


Block pilots



"comb" pilots

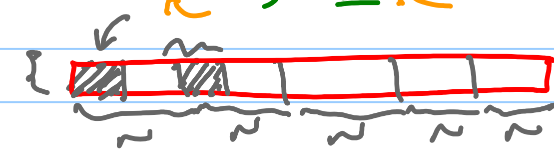




"Grid" pilots type

2D-CE (2D-LMMSE)

$\psi_t(\Delta t)$, $\psi_f(\Delta f)$



Freq. & Time Correlation

$h(\tau;t) \rightleftharpoons H(f;t)$

CIR ↗

$h(\tau,t) = \sum_{r=0}^{R} h_r(t)\,\delta(\tau-\tau_r)$ ↳ WSS-US

R, α, σ_τ^2

r^{th} paths are mutually independent

$H(f;t) = \int_{-\infty}^{+\infty} h(\tau;t)\, e^{-j2\pi f\tau}\, d\tau$ → STFT

$= \sum_r h_r(t)\, e^{-j2\pi f \tau_r}$

$R(\tau) = E[x(t)\,x^*(t-\tau)]$

2-D Correlation function

$\psi(\Delta t, \Delta f) = E\left[H(t+\Delta t, f+\Delta f)\, H^*(t,f) \right]$

$= \psi_t(\Delta t) \cdot \psi_f(\Delta f)$ → Since time & freq. "behaviour" of the channel are independent.

Doppler profile (under $\psi_t(\Delta t)$); pdp (under $\psi_f(\Delta f)$)

→ Jake's PSD || Laplacian PSD

Exercise #1 : Show that

(i) $\psi_f(\Delta f) = \sum_r \sigma_r^2 e^{-j2\pi \Delta f \tau_r}$; $\Delta f \to$ "freq. step"

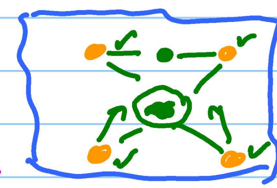
(ii) $\psi_t(\Delta t) = J_0(2\pi f_D \Delta t)$; $\Delta t \to$ "time step"

$f_D \to$ Doppler freq. $2V/\lambda$

$J_0(\cdot) \to$ zeroth order Bessel fn. of the 1st kind

2-D LMMSE CE

$\psi_f(6) \cdot \psi_t(2)$



$y_p = H_p x_p + v$, $v \sim \sigma_v^2$

$\frac{y_p}{x_p} = \hat{H}_{p,1}$ (zero-forcing CE)

$\hat{H}_{LMMSE} = W^H_{4\times 1} \hat{H}_p$ (4×1)

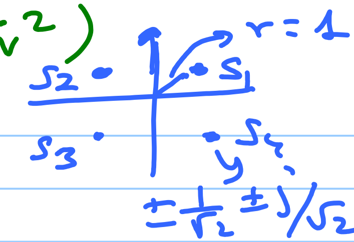
$E[\hat{H}\hat{H}^H]_{4\times 4}$

$\hat{H}_p = [\hat{H}_{p,1}, \hat{H}_{p,2}, \hat{H}_{p,3}, \hat{H}_{p,4}]^T$

$W = \left(R_{\hat{H}\hat{H}} + \sigma_v^2 I_{4\times 4}\right)^{-1}_{4\times 4} \cdot \overline{\hat{H} H^*}_{4\times 1}$

Real | y, H, I, v **Scalar** $\longrightarrow$ **Vector**

$$y = H I + v \rightarrow ①, \quad v \sim \mathcal{N}(0, \sigma_v^2), \quad I \leftarrow d$$



(i) Zero-forcing CE:

$$\hat{H}_{ZF} = \frac{y}{d} \longrightarrow d^* y = \hat{H}_{ZF}$$

linear, Gaussian

(ii) Least Squares CE:

$$\min_{\hat{H}} (y - \hat{y})^2, \quad \text{where } \hat{y} = \hat{H}_{LS} \cdot d$$

$$\frac{d}{dH}(y - \hat{y}) \longrightarrow 2(y - \hat{y})d = 0 \Rightarrow \hat{H}_{LS} = \frac{y}{d}$$

$I \rightarrow$ known

(iii) Maximum Likelihood CE:

$$\max_{\hat{H}} p(y|H) \leftarrow ①$$

v

$$p(y|H) = \frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-\frac{1}{2}(y - Hd)^2}$$

v^2

$v \rightarrow \mathcal{N}(0, \sigma_v^2)$

$\int \log_e(\cdot) \quad \min_H (y - Hd)^2 \longrightarrow \boxed{\hat{H}_{ML} = \frac{y}{d}}$;

MMSE = LMMSE $\rightarrow$ LMS

(iv) Linear MMSE (LMMSE) : $\min_W E\left[(H-\hat{H})^2\right]$ $\quad y = Hd + v$ (random variable)

where $\hat{H} = W y$ $\quad \sigma_d^2 = E[d^2]$

$\frac{d}{dw} E\left[(H-\hat{H})^2\right]$

$$W = \frac{E[Hy]}{E[y^2]} = \left(\frac{d\,E[H^2]}{(\sigma_d^2 E[H^2] + \sigma_v^2)}\, y \right) = \hat{H}$$

$d \rightarrow$

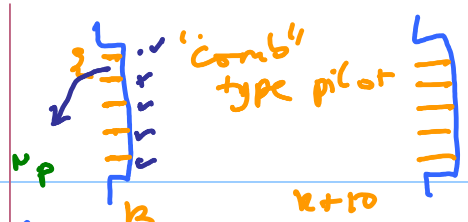
LMMSE, if $\sigma_v^2 << \sigma_d^2 \simeq \frac{d\,y}{\sigma_d^2}$

N_y

$\vdots\ y(n+1) = H(n+1)$

$\vdots\ y(n) = H(n)\, d + v$

$\vdots\ y(n-1) = H(n-1)$



1-D Channel Estimation

- ✓ Modified LS (mLS) ✓
- modified LMMSE ✗
- ✓ FFT-based CE ✓

$\rightarrow \bar{Y}_p = D \cdot \bar{H}_p + \bar{V}_p$

$N_p \times 1$ $N_p \times N_p$ $N_p \times 1$

(1) mLS

$N \rightarrow$ FFT size ; $\quad N_p \rightarrow$ # of pilot subs. in N_u

$N_{cp} \rightarrow$ Cyclic Prefix ;

$L \rightarrow$ CIR length $\quad L \leq N_{cp}-1$

$$D = \begin{bmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}$$

$$\bar{y}_p = D \cdot (F_p \, \bar{h}) + V_p$$

$N_{cp} \times 1$

Reduced DFT matrix

$\triangleq A_p$

$F_p \rightarrow \dim(N_p \times N_{cp})$ ✓

$$F = \left[\underbrace{\quad}_{N_{cp}} \quad \right]_{N \times N}$$

$F_p \rightarrow F_R$

N_p

N_{cp}

$\bar{y} = A_p \cdot \bar{h} + \bar{V}_p$

N_p Equ.

$\min_{\bar{h}_{LS}} \left| \bar{y}_p - \hat{\bar{y}}_p \right|^2 \qquad \hat{\bar{y}}_p \equiv A_p \, \bar{h}_{LS}$

$A_p = D F_p$

$$\bar{h}_{LS} = \big(\underbrace{A_p^H A_p}_{N_{cp} \times N_{cp}}\big)^{-1} A_p^H \, \bar{y}_p$$

$N_{cp} \times 1$

$N_p > N_{cp}$

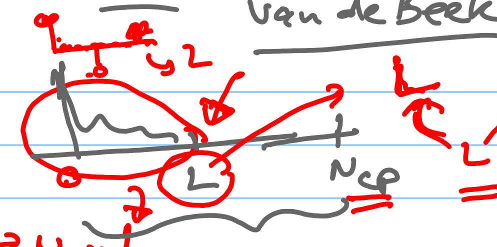
Exercise #2 : Show that

$$N_u \times 1 \rightarrow \hat{H}_{LS} = F_{N \times N} \, \bar{h}_{LS}$$

$N \left\{ \begin{bmatrix} \square \\ | \\ | \end{bmatrix} \right. \begin{matrix} \} N_{cp} \\ \} N-N_{cp} \end{matrix}$

(*) Matrix Regularisation ($A_p^H A_p + \varepsilon I_{N_{cp} \times N_{cp}})^{-1}$

L_0, L_1 ← "Sparse" ($\varepsilon \to 0$) ← $\}N_p$ → L_2 ✓ — van de Beek ✓



$\tilde{A}_p \to N_p \times L$

(2) M-LMMSE :

$$\bar{y}_p = D \bar{F}_p \bar{h} + \bar{v}_p \quad (1)$$

$$\min_{W} E\left[\,|\bar{h} - \hat{\bar{h}}|^2\right] \text{ where } \hat{\bar{h}} = W_2 \bar{y}$$

$\hat{h}$: $N_{cp} \times 1$; W_2: $N_{cp} \times N_p$; $\bar{y}$: $N_p \times 1$

$W \leftarrow N_{cp} \times N_p$

$$W = E[\bar{h}\bar{y}^H]\, E[\bar{y}\bar{y}^H]^{-1}$$

$E[\bar{h}\bar{y}^H]$: cross-corr. matrix → $N_{cp} \times N_p$

$E[\bar{y}\bar{y}^H]$ → $N_p \times N_p$

← Substitute for $\bar{y}$ from (1)

$$\hat{h}_{m\text{-}LMMSE} = W \cdot \bar{y}, \quad W = (R_{hh} F_p^H D^{\#}) (D F_p R_{hh} F_p^H D^{\#} + \sigma_v^2 I)^{-1}$$

W: $N_{cp} \times N_p$; $(\cdot)^{-1}$: $N_p \times N_p$

$$\hat{H} = F \hat{h} \quad [\;\_\_\_\_\;, \;\_\_\_\_\_\_\;]$$ ($N - N_{cp}$)

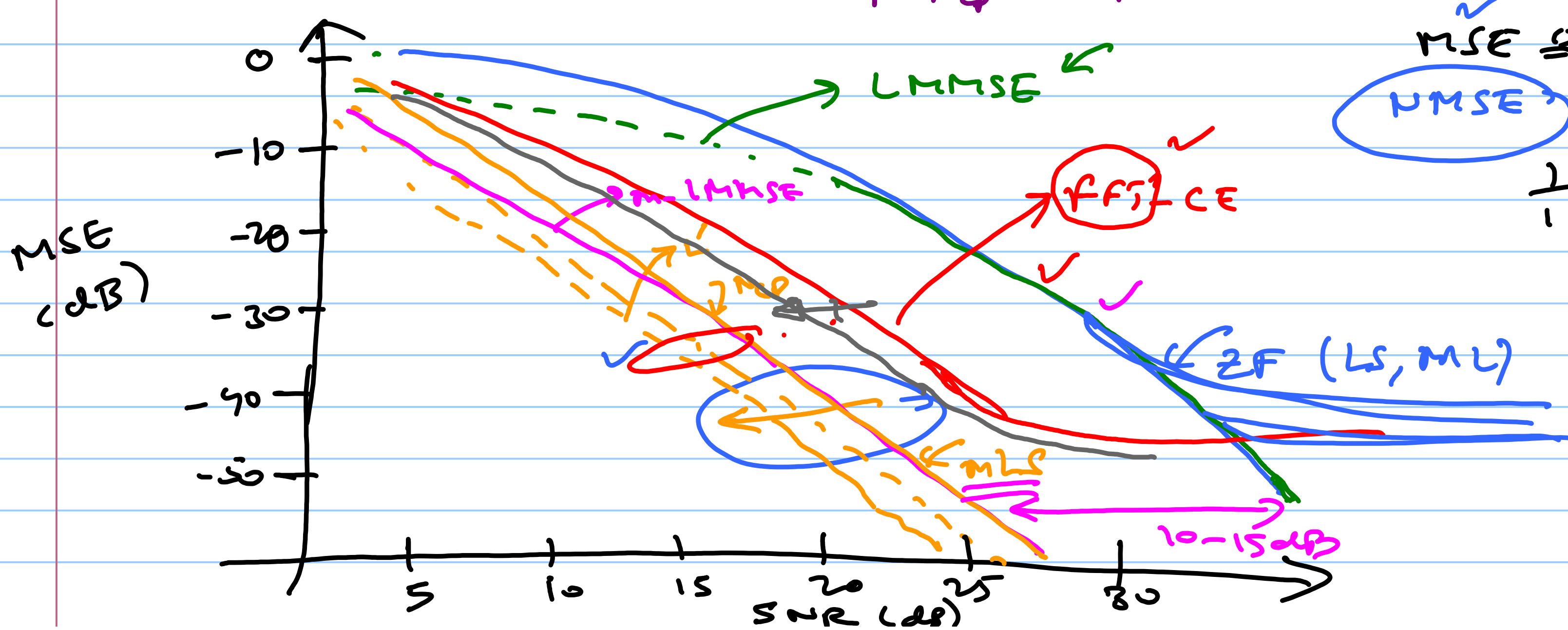
$$R_{hh} = E[\bar{h}\bar{h}^{\#}] = p \cdot dp \cdot$$ ($N_{cp} \times N_{cp}$)

$$MSE \triangleq E\left((H - \hat{H})^2\right)$$

NMSE

$$\frac{1}{100R} \sum_{i=1}^{100R} |\bar{H}_i - \hat{\bar{H}}_i|^2$$

mLS, ZF, mMMSE, LS, LMMSE



eg: $N = 1024$; $N_{cp} = 64$; $L \approx 30$

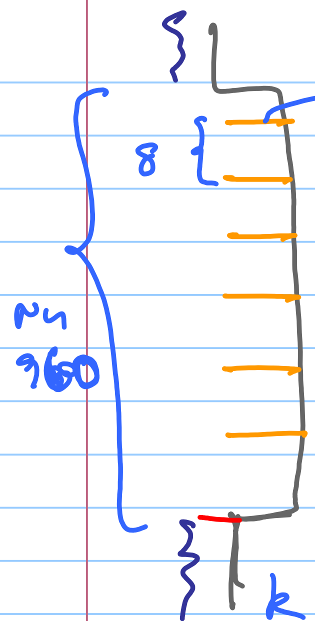
$d \in \sigma_d^2 = 1$

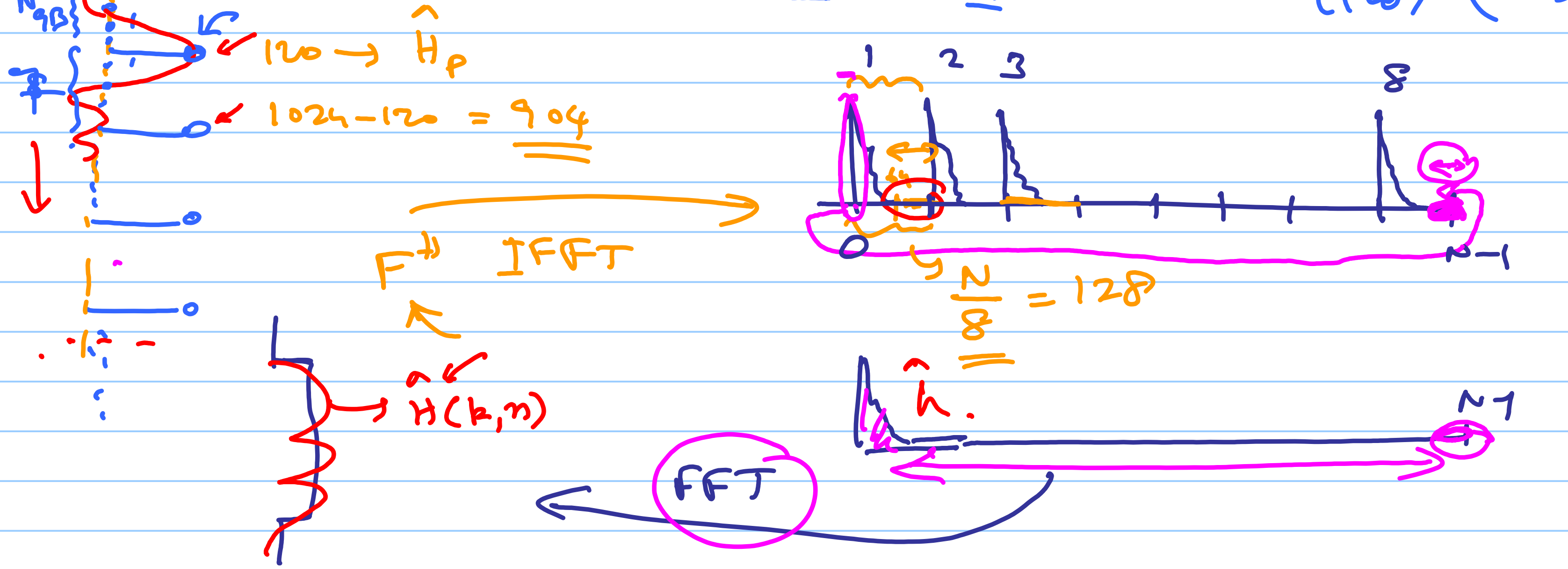
$\frac{960}{8} = 120$ pilots

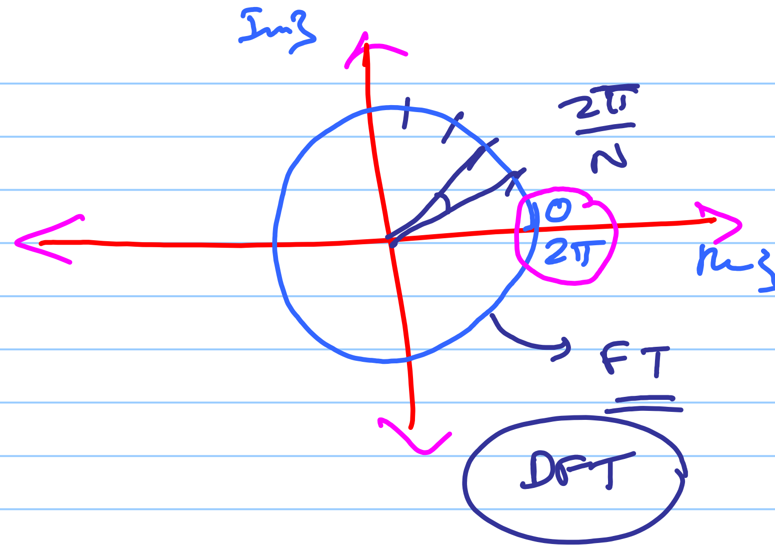
check $N_p > N_{cp}$ (120) (64)

$y_p = H_p \cdot d + v_p$

$y_p \cdot d_p^* = H_p + \tilde{v}_p$







Im{}
Re{}
2π/N
θ/2π
FT
=
DFT