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EE 6110: Adaptive Signal Processing

October 08 2021 **Tutorial #2** KG/IITM

Note: Questions marked with a " * " are perhaps of higher difficulty

1. If $\mathbf{A}^H = -\mathbf{A}$, (a) show that j**A** is Hermitian ($j=\sqrt{-1}$)
(b)* show that **A** is unitary, diagonalizable and has pure imaginary eigenvalues.

2. Find *k*, *l*, and *m* to make the matrix **A** shown below, a Hermitian matrix

$$A = \begin{bmatrix} -1 & k & -j \\ 3-j5 & 0 & m \\ l & 2+j4 & 2 \end{bmatrix}$$

3. In each of the below, find the unitary matrix **Q** that diagonalizes **A**, i.e., $\mathbf{Q}^H\mathbf{A}\mathbf{Q} = \mathbf{\Lambda}$.

a)
$$A = \begin{bmatrix} 4 & 1-j \\ 1+j & 5 \end{bmatrix}$$

b)
$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+j \\ 0 & -1-j & 0 \end{bmatrix}$$

c)
$$A = \begin{bmatrix} 2 & j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 2 & 0 \\ j/\sqrt{2} & 0 & 2 \end{bmatrix}$$

4. If **A** has $\lambda_1 = 0$, and $\lambda_2 = 5$ corresponding to

$$q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad q_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Find: (a) trace(**A**), (b) det(**A**), and
(c) Can you specify **A**?

5.* Find a real matrix **A** with $\mathbf{A} + \alpha\mathbf{I}$ invertible for all real α.

6. Find (i) $\mathbf{A}^{90}$, (ii) $e^{\mathbf{A}}$, if

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

7.* Find a 3 x 3 matrix whose rows add up to 1 and show that $\lambda = 1$ is an eigenvalue of this matrix. What is the corresponding **q**?

8. Verify if the matrices are unitary; if so, specify their inverses.

a)

$$A = \begin{bmatrix} -j/\sqrt{2} & j/\sqrt{6} & j/\sqrt{3} \\ 0 & -j/\sqrt{6} & j/\sqrt{3} \\ j/\sqrt{2} & j/\sqrt{6} & j/\sqrt{3} \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 3/5 & j4/5 \\ -4/5 & j3/5 \end{bmatrix}$$

c)

$$A = \frac{1}{4}\begin{bmatrix} j+\sqrt{3} & 1-j\sqrt{3} \\ 1+j\sqrt{3} & j-\sqrt{3} \end{bmatrix}$$

d)

$$A = \begin{bmatrix} \frac{1+j}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{j}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-j}{\sqrt{3}} \\ \frac{3+j}{2\sqrt{15}} & \frac{4+3j}{2\sqrt{15}} & \frac{5j}{2\sqrt{15}} \end{bmatrix}$$

9. Consider a recursion where $\mathbf{x}(0) = [\,2\ 0\ 2\,]^T$ and $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$

$$\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \end{bmatrix}$$

Find the (i) eigenvalues and the eigenvectors of the matrix **A**, (ii) the value of $\mathbf{x}(k+1)$, (iii) the limiting value $\mathbf{x}(\infty)$.

10.* A WSS process has $\mathbf{R}_{xx}(0) = 1$, and $\mathbf{R}_{xx}(\pm 1) = 0.8$.

a) Choose $\mathbf{R}_{xx}(\pm 2) \geq 0$, such that the process is deterministic like.

b) For your choice of $\mathbf{R}_{xx}(\pm 2)$, obtain an expression for $\mathbf{R}_{xx}(k)$ for all k.

11. Consider a r.p. $u(n) = 3e^{j4\pi n} + v(n)$, where v(n) is a Gaussian white noise process with variance $\sigma^2 = 4$.

a) find the 2 x 2 autocorrelation matrix **R.**

b) what is the eigenvalue spread of **R**?

c) find the expression for $\mathbf{R}^6$.

12. We are given random samples { $x_1, x_2, \ldots, x_N$} where each x_i is i.i.d. with $N(\mu,\sigma^2)$. Consider the following estimator for μ,

$$\hat{\mu}(N) = \frac{1}{N+a}\sum_{i=1}^{N} x_i$$

where $a \geq 0$.

(a) For what value(s) of a is the above unbiased estimator of μ ? (for small sample case)

(b) For what value(s) of a is the above an <u>asymptotically</u> unbiased estimator of μ ?

(c)* Prove that the above is a consistent estimator of μ, for all $a \geq 0$.

13. Suppose that N independent observations $\{x_1, x_2, \ldots, x_N\}$ are made of a r.v. X that is Gaussian; i.e.,

$$p(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-(x_i - \mu)^2 / 2\sigma^2]$$

Assuming only μ is unknown, derive the Cramer-Rao lower bound (CRLB) of $E[(\mu - \hat{\mu})^2]$ for an unbiased estimator of μ.

14.* For the observations in Pbm.13, now assume that only variance σ^2 is unknown, and derive the CRLB of $E[(\sigma^2 - \hat{\sigma}^2)^2]$ for an unbiased estimator of σ^2.