Note Title 27-03-2020

EE6110 Aug.-Nov., 2021

Class #23 & #24 (Oct. 28 & 29, 2021)

T SS n

Excess MSE (EMSE) → RLS @ steady state ($n \to \infty$)

Recall:

$$P(n) = \lambda^{-1}\left[P(n-1) - \frac{\lambda^{-1} P(n-1)\, \bar{u}(n)\, \bar{u}^H(n)\, P(n-1)}{1 + \lambda^{-1}\bar{u}^H(n)\, P(n-1)\, \bar{u}(n)} \right]$$

$$= \lambda^{-1} P(n-1)\left[I - \frac{\lambda^{-1}(P u^H u P)}{(\cdot)} \right]$$

Hermitian

stabilized RLS

$\bar{u}^H \bar{u}$ ← Euclidean distance $\|\bar{u}\|^2$

$\bar{u}^H P \bar{u}$ → $\|\bar{u}\|^2_{P(n)}$

Square weighted Euclidean Norm

&

Subtract from both sides:

$\bar{w}_o$ or $\bar{w}_o - ($

$$\underbrace{\bar{w}_o - w(n)}_{\tilde{w}(n)} = w(n-1) + P(n)\underbrace{\left[d(n) - \bar{w}^H(n-1)\,\bar{u}(n) \right]}_{e^*(n)} \bar{u}(n))$$

$$\boxed{\tilde{w}(n) = \tilde{w}(n-1) - p(n)\, e^*(n)\, \bar{u}(n)} \rightarrow (1)$$

→ Multiply both sides by $\bar{u}^H(n)$

$$\boxed{e_p(n) = e_a(n) - \underbrace{\bar{u}^H(n)\, P(n)\, \bar{u}(n)}_{\triangleq \|\bar{u}(n)\|^2_{P(n)}} e^*(n)} \rightarrow (2)$$

a-posteriori error

a-priori error

$e^*(n)$

$$\frac{e_a - e_p}{\|u\|^2_p}$$

→ when $\bar{u}(n) \neq 0$, eliminate $e^*(n)$ in (1) by using (2)

$$\tilde{w}(n) = \tilde{w}(n-1) - p(n)\left(\frac{e_a - e_p}{\|u\|^2_p}\right) \cdot \bar{u}(n)$$

$$\Rightarrow \quad \tilde{w}(n) + \frac{P(n)\,\bar{u}(n)}{\|\bar{u}\|^2_{P(n)}}\, e_a(n) = \tilde{w}(n-1) + \frac{P(n)\,\bar{u}(n)}{\|\bar{u}(n)\|^2_{P(n)}}\, e_p(n)$$

$\bar{u}(n)$

$$\checkmark \quad \bar{a} + \bar{b} = \bar{c} + \bar{d}$$

<u>Exercise</u>:

using $P^{-1}(n)$ as weighting matrix and equating the squared weighted norms

$$\underbrace{(\bar{a}+\bar{b})^H P^{-1} (\bar{a}+\bar{b})} = (\bar{c}+\bar{d})^H P^{-1} (\bar{c}+\bar{d})$$

$$P^{-1} \rightarrow P^{-1}(n)$$

$$\boxed{\|\tilde{w}(n)\|^2_{P^{-1}(n)} + \bar{\mu}(n)\,|e_a(n)|^2 = \|\tilde{w}(n-1)\|^2_{P^{-1}(n)} + \bar{\mu}(n)\,|e_p(n)|^2} \quad (3)$$

$$\mu \;\; 1/\|\bar{u}(n)\|^2_{P(n)}, \text{ if } \bar{u}(n) \neq 0$$

check that cross-terms cancel out

where $\bar{\mu}(n) = \begin{cases} 0 & , \text{otherwise.} \end{cases}$

Since at steady state,

$$E\left[\tilde{w}(n)\tilde{w}^H(n)\right] = E\left[\tilde{w}(n-1)\tilde{w}^H(n-1)\right] = C_{M\times M}$$

$n \to \infty$

$(P^{-1}(n))$

$P(n) \longrightarrow \bar{u}(n), \quad \bar{u}(j), \; j \leq n$

$P(n) \longrightarrow$ | mean values

$P(n-1) \longrightarrow$ | mean values

Steady State Approximation

<u>Recall</u> : $\Phi(n)$, $P^{-1}(n) = E \sum_{j=1}^{n} \lambda^{n-i}\, \bar{u}(j)\,\bar{u}(j)^H + \lambda^n \delta I_{M\times M}$

E

As $n \to \infty$, and for $\lambda < 1$,

$$\lim_{n\to\infty} E\left[P^{-1}(n)\right] = R_{uu} \cdot \underbrace{\sum_{j=i}^{\cancel{n}\ \infty} \lambda^{n-i}}_{\frac{1}{1-\lambda}} + \underbrace{\lambda^n \delta I_{M\times M}}_{=0}$$

$$\lim_{n\to\infty} E\left[P^{-1}(n)\right] \triangleq R_{uu}\left(\frac{1}{1-\lambda}\right) \triangleq P^{-1}$$

Note: The mean value of $P(n)$ is difficult to evaluate

$$E\left[P(n)\right] \approx \left\{E\left[P^{-1}(n)\right]\right\}^{-1} = \boxed{(1-\lambda) R_{uu}^{-1} = P}_{n\to\infty}$$

eg: $x \to [1,2] \longrightarrow E[x] = 1.5$

uniform RV

$$\frac{1}{1.5} = \frac{2}{3} = 0.66\bar{6}$$

$g(x) \rightarrow y = \frac{1}{x}$; $E[y] = ?$ $\qquad E[g(x)] = \int_1^2 g(x) f_x(x)\, dx$ $\quad f_x \to 1$

$$= \int_1^2 \frac{1}{x} \cdot 1 \cdot dx = \ln(x)\Big|_1^2 = \ln(2) - \cancel{\ln(1)}^{\,0}$$

$$= 0.693$$

Returning to (3), at steady state

$$E\left[\|\tilde{w}(n)\|_{P^{-1}(n)}\right] \approx E\left[\|\tilde{w}(n)\|_{P^{-1}}\right] = Tr\left(C P^{-1}\right)$$

Note: $\bar{x}^H \bar{x} = Tr\left[\bar{x}\bar{x}^H\right]$

$(P^{-1})^H = P^{-1}$

$$E\left(\tilde{w}^H(n)\,P^{-1}\tilde{w}(n)\right) = E\left(Tr\left[\tilde{w}(n)\,\tilde{w}^H(n)\,(P^{-1})^H\right]\right)$$

$$= Tr\left[\underbrace{E\left[\tilde{w}(n)\,\tilde{w}^H(n)\right]}_{C}\,P^{-1}\right]$$

$$\Rightarrow E\left[\|\tilde{w}(n)\|_{P^{-1}(n)}\right] = Tr\left(C P^{-1}\right) = E\left[\|\tilde{w}(n-1)\|_{P^{-1}(n-1)}\right]$$

Putting this in (3)

$$E\left[\bar{\mu}(n)\,|e_a(n)|^2\right] = E\left[\bar{\mu}(n)\,|e_p(n)|^2\right] \quad \text{as } n\to\infty$$

$e(n) = d(n) - \bar{w}^H(n-1)\,\bar{u}(n)$

$e_a(n) = (w_o - \bar{w}(n-1))^H\,\bar{u}(n)$

$(w_o - \bar{w}(n))^H\,\bar{u}(n)$

$e_p(n) =$ $\hat{d}(n)$

MSE

EMSE

$e(n) = e_a(n) + v(n) \rightarrow \xi^{LMS} = E\left[|e(n)|^2\right]$

using eq. (2), to eliminate $e_p(n)$

$\bar{\mu} = \begin{cases} \frac{1}{\|u(n)\|^2_P}, & u(n) \neq 0 \\ 0, & \ldots \end{cases}$

$$E\left[\bar{\mu}(n)\,|e_a(n)|^2\right] = E\left[\bar{\mu}(n)\left|e_a(n) - \|\bar{u}(n)\|^2_{P(n)}\, e(n)\right|^2\right] \quad n \to \infty$$

Exercise: show this

$$\boxed{E\left[\|u(n)\|^2_{P(n)}\,|e(n)|^2\right] = 2\,\mathrm{Re}\left\{E\left[e_a^*(n)\,e(n)\right]\right\} \quad n \to \infty}$$

Variance relation for RLS

(4)

Since $\{ d(n), \bar{u}(n) \}$ satisfy the LRM

putting this in ④ $\longleftarrow$ $d(n) = \bar{w}_0^H \bar{u}(n) + v(n)$ & $\boxed{e(n) = e_a(n) + v(n)}$

$E[v(n)] = 0$ uncorrelated

$$\sigma_v^2 \, E\left[\|\bar{u}(n)\|^2_{P(n)} \right] + E\left[\|\bar{u}(n)\|^2_{P(n)} \, |e_a(n)|^2 \right] = 2\, E\left[\underbrace{|e_a(n)|^2}_{} \right]$$

$2\,\xi^{RLS}$

Using the "Separation Principle"

i.e., assume that $\|\bar{u}(n)\|^2_{P(n)}$ is independent (orthogonal to) of $e_a(n)$

$$E\left[\|\bar{u}(n)\|^2_{P(n)} \, |e_a(n)|^2 \right] = E\left[\|\bar{u}(n)\|^2_{P(n)} \right] \cdot E\left[|e_a(n)|^2 \right]$$

$\|x\|^2_{P} = Tr[x x^H]$

Assume $\lim_{n\to\infty}$, we replace $P(n)_{m\times m}$ by its mean value $P_{m\times m}$ ✓ $\swarrow$ ξ_{RLS}

$$E\left[\|\bar{u}(n)\|_P^2\right] = E\left[\bar{u}^H P \bar{u}\right] = E\left[\mathrm{Tr}\left[\bar{u}\bar{u}^H P\right]\right]$$

$$= \mathrm{Tr}\left[E\left[\bar{u}\bar{u}^H\right] P\right]$$

$$= \mathrm{Tr}\left[R_u \cdot P\right] \checkmark$$

Recall $(1-\lambda) R_u^{-1} = P$

$$\Rightarrow \mathrm{Tr}\left[R_u P\right] = \mathrm{Tr}\left[(1-\lambda)\cdot I_{m\times m}\right] = M\cdot(1-\lambda) = E\left[\|\bar{u}(n)\|_P^2\right]$$

$$\text{|||}^{ly} \quad \sigma_v^2\, E\left[\|\bar{u}(n)\|_{P(n)}^2\right] = \sigma_v^2 \cdot (1-\lambda)\cdot M$$

$$\therefore \quad \sigma_v^2 (1-\lambda) M + M(1-\lambda)\xi^{RLS} = 2\xi^{RLS}$$

$$\Rightarrow \boxed{\xi^{RLS} = \frac{\sigma_v^2 (1-\lambda) M}{2-(1-\lambda)M}}$$

<u>Recall</u>

$$\xi^{\epsilon\text{-NLMS}} = \frac{\mu \cdot \sigma_v^2}{2-\mu}, \quad \epsilon \to \text{small}$$

$$\approx \frac{\sigma_v^2 (1-\lambda) M}{2}$$

$\lambda \approx 1$ $1-0.999 = 0.001 \rightarrow \underline{\underline{0.01}}$

✓

$M \times 1$ $M \times M$

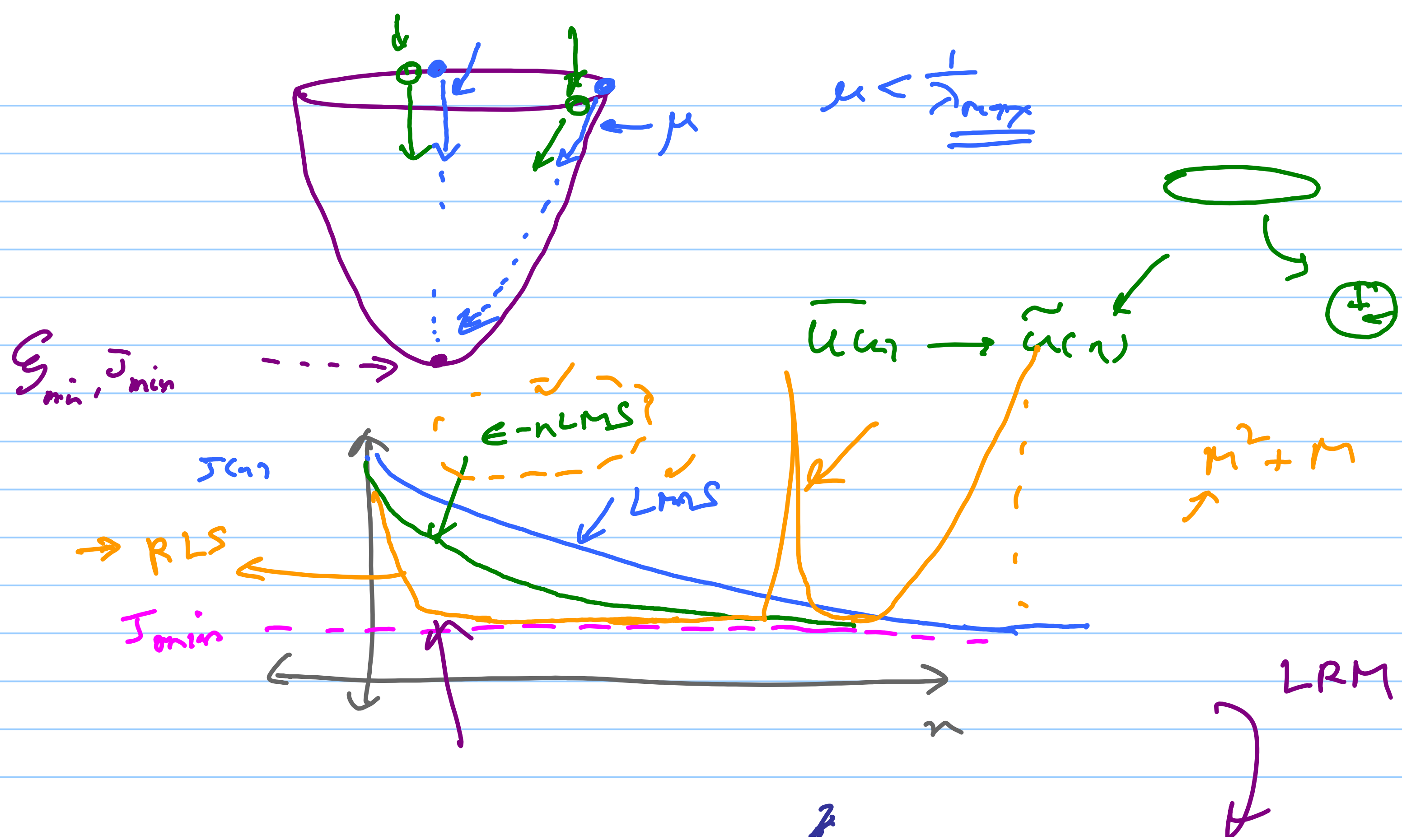
$M \times 1$, $M^2 \times M$

$\lambda = 0.999$ $\Longrightarrow$ $\underbrace{(1-\lambda)M}_{}$ $\longleftrightarrow$ μ

RLS EMSE ϵ-NLMS

EMSE

$\mu \to$ smaller



μ
$\mu < \frac{1}{\lambda_{max}}$
ξ_{min}, J_{min}
ϵ-nLMS
$J(n)$
LMS
RLS
J_{min}
n
$\bar{u}(n) \rightarrow \tilde{u}(n)$
$M^2 + M$
LRM

LMMSE Weiner filter $\longrightarrow$ Newton Algo $\longrightarrow$ Steepest Descent $\longrightarrow$ LMS $\longrightarrow$ RLS

n-LMS

LPM

$u(n) \longleftrightarrow d(n)$

$\bar{R}, \bar{P}$

Recursive

$\lambda \longrightarrow$ "forgetting factor"

$\bar{w}_0$

$\bar{w}_0 \rightarrow \bar{w}_0(n)$

ERM $\longleftrightarrow$ dynamic model

Recursive LMMSE ||

Kalman filter