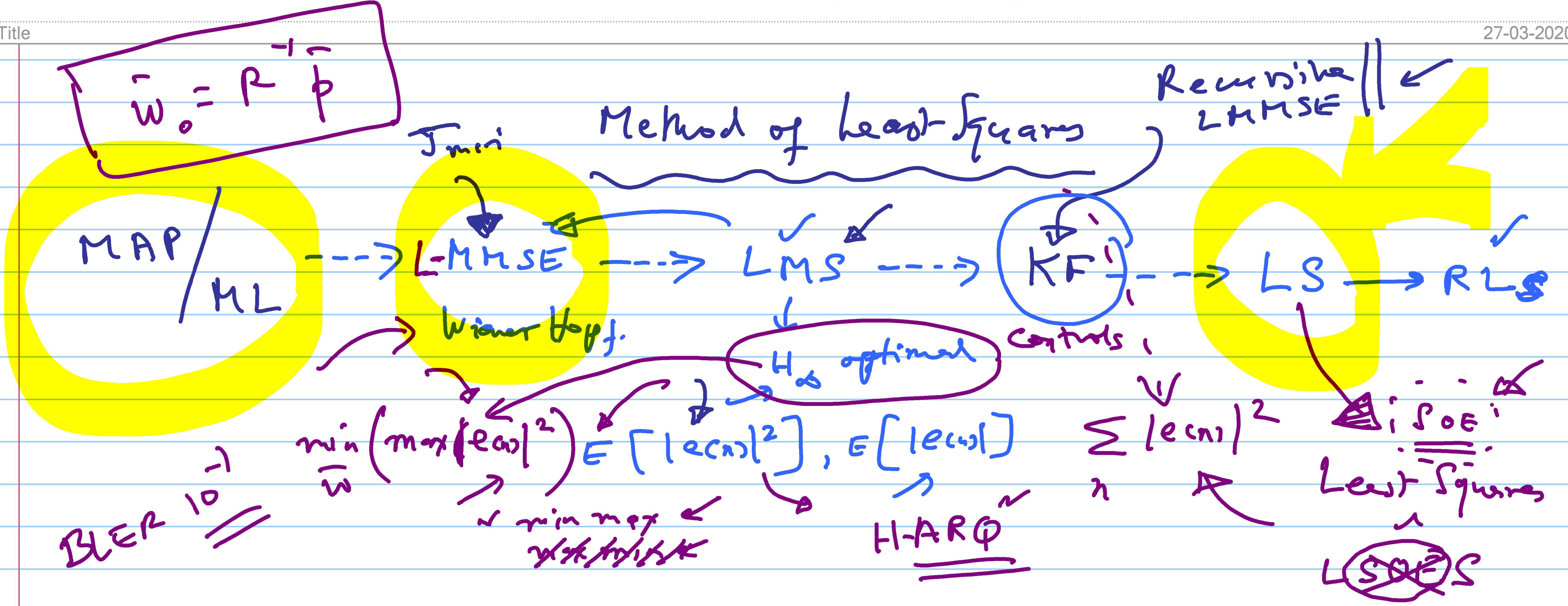
EE6110 Aug.-Nov., 2021

Class #18 and #19 (Oct. 09, 2021)

Note Title 27-03-2020



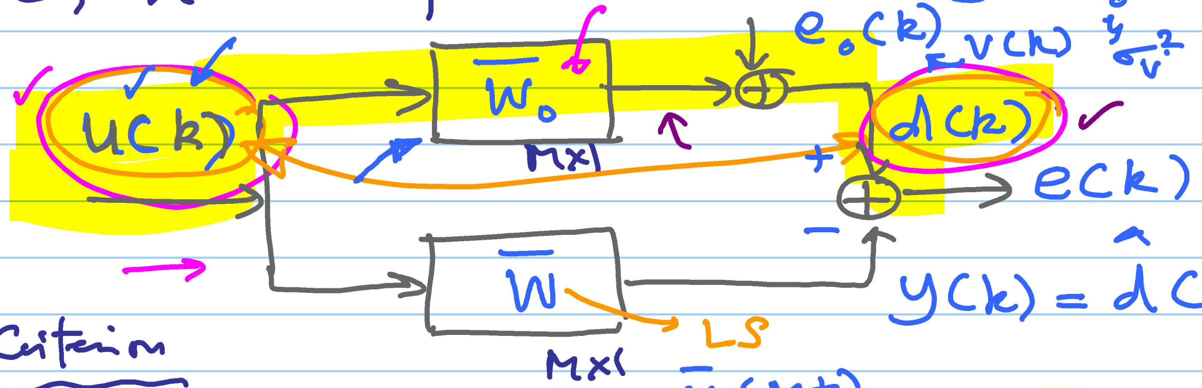
Recall : Linear LS Estimation Model ←

↳ Linear Regression Model

$\sigma_{e_0}^2 = J_{min}$ & σ_v^2

$\bar{w}_0 \to$!!

R, $\bar{p}$



$y(k) = \hat{d}(k)$

N ↔ $\underline{\underline{M}}$

$\underline{\underline{N >> M}}$

LS Criterion

$\bar{u}(M+1)$

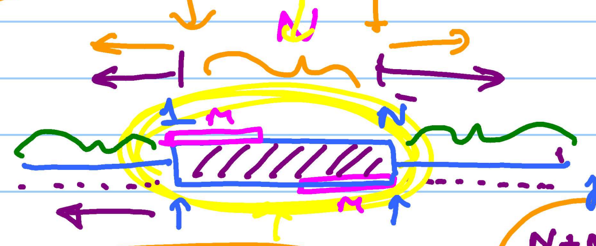
$$\text{Given } \{\underbrace{u(1), u(2), u(3), \ldots u(M)}_{\bar{u}(M)}, u(M+1), \overbrace{\ldots u(N-1), u(N)}^{\bar{u}(N)}\}$$

(N−M+1)

$$\min_{\bar{W}_{M\times 1}} \sum_{i=M}^{N} |e(i)|^2$$

← Covariance formulation

$\sum_{1}^{N}$



$\hat{d} \to y$

$d - \hat{d} = e$

Auto-corr. formulation → $\sum_{1}^{N+M}$

N+M

$\sum_{M}^{N+M}$

← Pre-win. , Post-win

Matrix - vector Notation

LS → RLS → KF

Define : $A^H_{M\times(N-M+1)} \triangleq \overset{M}{\updownarrow}\underbrace{\begin{bmatrix} | & | & & | \\ \bar{u}(M) & \bar{u}(M+1) & \cdots & \bar{u}(N) \\ | & | & & | \end{bmatrix}}_{\longleftarrow N-M+1 \longrightarrow}$; $A = \begin{bmatrix} - \bar{u}^*(M) - \\ - \bar{u}^*(M+1) - \\ \vdots \\ - \bar{u}^*(N) - \end{bmatrix}$;

$\bar{d}^H = [\overbrace{d(M)\ \ d(M+1) \cdots d(N)}^{N-M+1}]$; $\bar{d} = \begin{bmatrix} d^*(M) \\ \vdots \\ d^*(N) \end{bmatrix}$; $\boxed{\bar{d} = A\bar{w}_0 + \bar{e}_0}$

$\boxed{\bar{y} = A\bar{w}}$

$\min_{\bar{w}_{M\times 1}} \sum_{i=M}^{N} |e(i)|^2$ ← Exercise : Show that this leads to → $\boxed{\Phi\ \bar{w} = \bar{z}}$

$\frac{d}{d\bar{w}}(\Sigma|\cdot|^2) = 0$

Recall : wiener hopf.

$R\bar{w} = \bar{P}$

where $\Phi_{M\times M} = A^H A_{M\times M}$; $\bar{z}_{M\times 1} = A^H_{M\times(N-M+1)} \bar{d}_{(N-M+1)\times 1}$; $\bar{w}_{LS} = \Phi^{-1}\bar{z}$

$$\bar{w}_{LS} = \Phi^{-1}\bar{z} = \underbrace{(A^H A)^{-1} A^H}_{A^{\dagger} \mapsto \text{"perp"}} \bar{z}$$

Moore-Penrose Pseudo Inverse

(*) Principle of orthogonality (in LS)

$$\to E[e(k)\,\bar{u}(k)] = \bar{0}$$

$$\to \sum_{k=M}^{N} y(k)\, e^{*}_{min}(k) = 0 \checkmark$$

$N-M+1$

$$e(k) = d(k) - \hat{d}(k)$$

$$= d(k) - y(k)$$

$$\Rightarrow d(k) = e_{min}(k) + y(k)$$

$$\bar{d}^H \bar{d};$$

Show that

$$J \dashrightarrow \xi_d = \xi_{min} + \xi_{est}$$

$$\xi_{est} = \bar{y}^H \bar{y}$$

$$\bar{y} = A\bar{w}$$

$$\bar{y}^H = \bar{w}^H A^H$$

$$= \bar{w}^H \underbrace{(A^H A)}_{\Phi} \bar{w}$$

$$\Phi \bar{w} = \bar{z}$$

$$= \bar{w}^H \bar{z};$$

$$\xi_{min}^{\checkmark} = \xi_d - \xi_{est} = \bar{d}^H \bar{d} - \bar{w}_{LS} \Phi \bar{w}_{LS} ; \checkmark$$

$$= \bar{d}^H \bar{d} - \bar{w}_{LS} \bar{z} ; \checkmark \quad J_{min} = \sigma_d^2 - \bar{w}^H R \bar{w} = \sigma_d^2 - \bar{w}^H \bar{p}$$

Properties of the LS Estimate

stochastic

(1) In $\bar{d} = A\bar{w}_0 + \bar{e}_o(n)$, if $e_o(n)$ is zero mean $\Rightarrow$ $\bar{w}_{LS}$ is unbiased $\Rightarrow$ $E[\bar{w}_{LS}] = \bar{w}_0$;

Proof: $\bar{w}_{LS} = (A^H A)^{-1} A^H \bar{d} = \bar{w}_0 + (A^H A)^{-1} A^H \bar{e}_0(n)$

(2) If $e_o(n)$ is zero-mean & white, with variance σ^2

Exercise

Then $\mathrm{Cov}(\bar{w}_{LS}) \overset{\Delta}{=} E\left[(\bar{w}_{LS}-\bar{w}_o)(\bar{w}_{LS}-\bar{w}_o)^H\right] = \sigma^2 \cdot \Phi^{-1}$

BLUE

(3) If $e_o(n)$ is zero-mean, white, & Gaussian, then (efficient) $\bar{w}_{LS}$ achieves the Cramer-Rao LB for unbiased estimators.

$$\boxed{\underbrace{E\left[(\bar{w}_{LS}-\bar{w}_o)(\bar{w}_{LS}-\bar{w}_o)^H\right]}_{M\times M} \geq \underbrace{J^{-1}}_{M\times M}}$$

where J is the Fischer information matrix

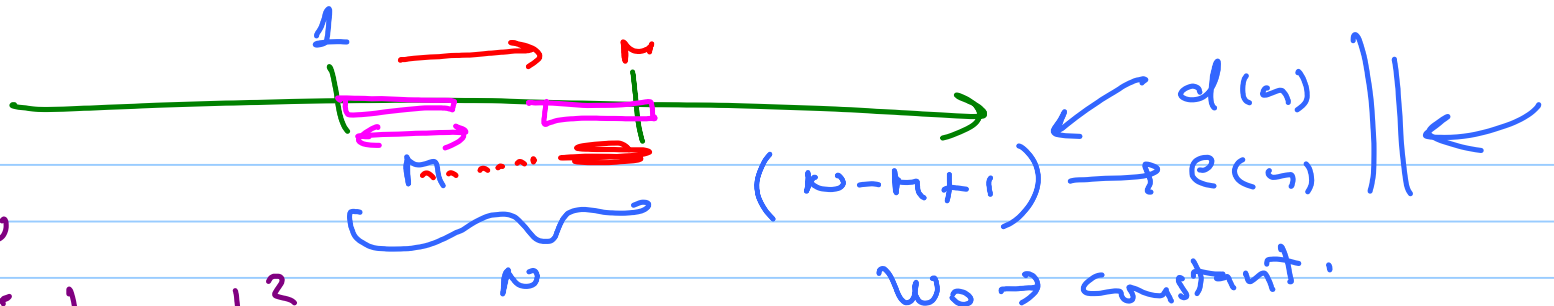
equality

$$\underset{M\times M}{J} \triangleq E\left[\left(\frac{\partial l}{\partial \bar{w}_0^*}\right)\cdot\left(\frac{\partial l}{d\bar{w}_0^T}\right)\right]$$

where $l = \ln f_{\bar{E}}(\bar{e}_0)$ where $f_{\bar{E}}(\bar{e}_0) = \frac{1}{(\pi\sigma^2)^{N-M+1}}$.

$$\exp\left[-\frac{1}{\sigma^2}\bar{e}_0^H \bar{e}_0\right]$$

Proof : First show that $J = \frac{1}{\sigma^2}\Phi_{M\times M}$



$w_0 \rightarrow$ constant.

$w_0(k)$ ..

$$\xi = \sum_{i=M}^{N} |e(i)|^2$$

"forgetting factor"

$$\xi = \sum_{i=1}^{n} \lambda^{n-i} |e(i)|^2 + \delta \lambda^{n} \|\bar{w}(n)\|^2$$

W-LS

Regularisation term