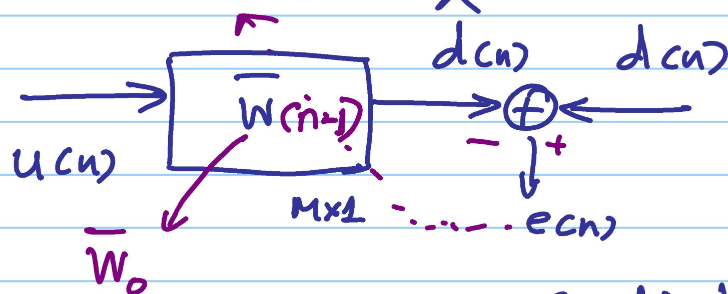
EE6110 Aug.-Nov., 2021

Class #15 & Class #16 (Sep. 30, 2021)

Note Title 27-03-2020

Recall



$E[|e(n)|^2]$

$\bar{w}_o = R^{-1}\bar{p}$

$\underset{M\times M}{R} = E[\bar{u}\bar{u}^H]$;

$\underset{M\times 1}{\bar{p}} = E[d^*(n)\bar{u}]$;

$$\bar{w}(n) = \bar{w}(n-1) + \mu \overbrace{E[e^*(n)\bar{u}(n)]}^{\text{gradient}}$$

$$\underbrace{\phantom{E[e^*(n)\bar{u}(n)]}}_{\bar{p} - R\bar{w}(n-1)}$$

$w(n) \rightarrow w_o$ $\left(\frac{1}{2}\right)$ $2/\lambda_{max}$

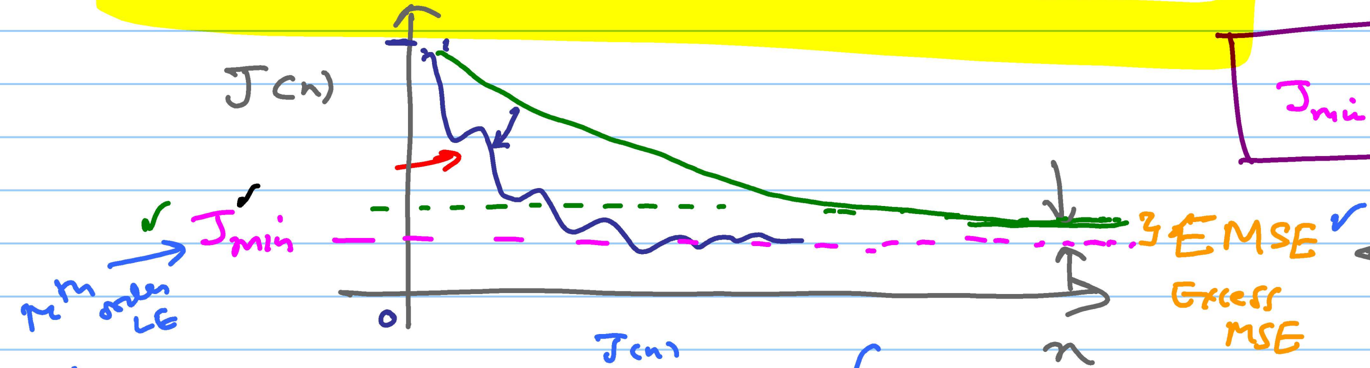
SDA $\rightarrow$ $\boxed{0 < \mu < \frac{1}{\lambda_{max}}}$

stochastic gradient

$$\bar{w}(n) = \bar{w}(n-1) + \mu \text{ "} e^*(n)\, \bar{u}(n) \text{"} \longrightarrow \text{LMS} \quad \mu = ?$$

$$e(n) = d(n) - \bar{w}^H(n-1)\, \bar{u}(n)$$

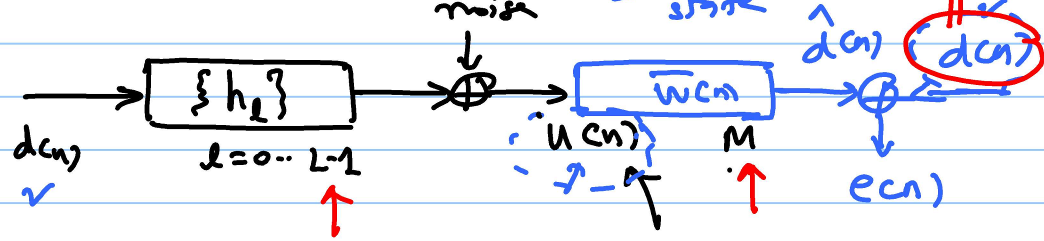
$$J_{min} = \sigma_d^2 - \bar{w}_0^H \bar{p}$$



$$\text{EMSE} \triangleq \lim_{n\to\infty} E\left[|e(n)|^2\right] - J_{min}$$

$J(n)$ — steady state

Recall: inv. filtering pbm:



$J(n) \longrightarrow J_{min}^*(\sigma_w^2, M, \Delta)$

"decoding delay"

Recall from the orthogonality condition

$$E\left[e^*(n)\,\bar{u}(n)\right] = 0$$

$$e(n) = d(n) - \bar{w}_o^H \bar{u}(n)$$

estimation error $\rightarrow$ $v(n)$

this suggest a linear regression model to relate $d(n)$ & $\bar{u}(n)$

$$\boxed{d(n) = \bar{w}_o^H \bar{u}(n) + v(n)}$$

$J_{min} = \sigma_d^2 - \bar{w}_o^H \bar{p}$

$= \sigma_v^2$

$J_{min} = \sigma_v^2$

$v(n)$ is independ of $\bar{u}(n)$

$E\left[v(n)\,\bar{u}(n)\right] = \bar{0}_{M\times 1}$

a-prior error

a-posteriori error

* Assumptions on $\{d(n), \bar{u}(n)\}$

- Initial condition $w(-1)$ is independent of $\{d(n), \bar{u}(n), v(n)\}$

$M\times 1$

- $R_{uu} = E\left[\bar{u}\bar{u}^H\right]$ is positive definite

$\bar{u} \rightarrow 1\times M$

- The rvs $\{d(n), \bar{u}(n), v(n)\}$ are zero mean $|\bar{v}, \bar{w}|$

Some Independence Results :

$g[x] = x$ $\quad \cancel{g[\bar{u}(n)]}$

In general $\quad \bar{w}_o -$

$$\underbrace{\bar{w}_o - \bar{w}(n)}_{\tilde{w}(n)} = \bar{w}_o - \Big(\underbrace{\bar{w}(n-1)}_{\tilde{w}(n-1)} + \mu\, g[e^*(n)]\, \bar{u}(n)\Big)$$

$$\boxed{\tilde{w}(n) = \tilde{w}(n-1) - \mu\, g[e^*(n)]\, \bar{u}(n)} \quad (1)$$

note

multiply both sides by $\bar{u}^H(n)$

$$\underbrace{\bar{u}^H(n)\,\tilde{w}(n)}_{\text{a-posteriori error}} = \underbrace{\bar{u}^H(n)\,\tilde{w}(n-1)}_{\text{a-priori error}} - \mu\, g[e^*(n)]\, \overbrace{\bar{u}^H(n)\,\bar{u}(n)}^{\|\bar{u}(n)\|^2}$$

$e(n) = d(n) - \hat{d}(n)$

$e_p(n)$ ✓ $\uparrow$

$e_a(n)$ ✓ $\uparrow$

$= \underline{\underline{d(n)}} - \underbrace{\bar{w}^H(n-1)\,\bar{u}(n)}$

a-priori

output estimation error

$$\therefore \quad \boxed{e_p(n) = e_a(n) - \mu\,\|\bar{u}(n)\|^2\, g[e^*(n)]} \quad (1a)$$

$$\Rightarrow \quad \boxed{g[e^*(n)] = \frac{1}{\mu\,\|\bar{u}(n)\|^2}\left(e_a(n) - e_p(n)\right);}$$

Fundamental Energy Conservation Equation

$$\tilde{w}(n) = \tilde{w}(n-1) - \left[e_a(n) - e_p(n)\right]\cancel{\mu} \cdot \bar{u}(n)$$

$$\cancel{\mu}\cdot\|\bar{u}(n)\|$$

$$\boxed{\tilde{w}(n)^{\,1} + \frac{e_q(n)}{\|\bar{u}(n)\|^2}{}^{\,2}\cdot \underset{M\times 1}{\bar{u}(n)} = \tilde{w}(n-1)^{\,3} + \frac{e_p(n)}{\|\bar{u}(n)\|^2}{}^{\,4}\,\bar{u}(n)}$$

$M\times 1$

Energy conservation eqn.

Taking the variance on each side $\rightarrow (\bar{a}^H\bar{a})$

$$\|\tilde{w}(n)\|^2 + \frac{|e_q(n)|^2}{\|\bar{u}(n)\|^2} = \|\tilde{w}(n-1)\|^2 + \frac{|e_p(n)|^2}{\|\bar{u}(n)\|^2}$$

<u>Exercise</u> : Show that the "cross-terms" cancel !!

Define $\vec{\bar{\mu}}(n) = \left(\|\bar{u}(n)\|^2\right)^{\dagger}$ ← pseudo-inverse

$$= \begin{cases} 1/\|\bar{u}(n)\|^2, & \text{if } \|\bar{u}(n)\| \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{\|\tilde{w}(n)\|^2 + \bar{\mu}(n)\,|e_a(n)|^2 = \|\tilde{w}(n-1)\|^2 + \bar{\mu}(n)\,|e_p(n)|^2}$$

2

E wrt $d(n), \bar{u}(n)$

Steady State filter operation :

const.

(*) $E[\tilde{w}(n)] = E[\tilde{w}(n-1)] = \bar{s}$, $s \cdot \bar{1}_{M\times 1}$ $n \to \infty$

(*) $E[\tilde{w}(n)\,\tilde{w}^H(n)]_{M\times M} = E[\tilde{w}(n-1)\,\tilde{w}^H(n-1)] = C_{M\times M}$, $n \to \infty$

(*) $E[\|\tilde{w}(n)\|^2] = E[\|\tilde{w}(n-1)\|^2] = Tr(C)$

Recollect, $\text{EMSE} \triangleq \lim_{n\to\infty} E\left[|e(n)|^2\right] - J_{min}$

Using the linear regression model

output est. error

$$e(n) = d(n) - \bar{w}^H(n-1)\,\bar{u}(n) \qquad J_{min} = \sigma_v^2$$

$$= w_o^H \bar{u}(n) + v(n) - \tilde{w}^H(n-1)\bar{u}(n)$$

$$\boxed{e(n) = e_a(n) + v(n)}$$

3

$$\lim_{n\to\infty} E\left[|e(n)|^2\right] = \lim_{n\to\infty} E\left[|e_a(n)|^2\right] + \sigma_v^2$$

$$\Rightarrow \xi \quad \boxed{\text{EMSE} = \lim_{n\to\infty} E\left[|e_a(n)|^2\right]}$$

Variance Relation for Steady State Performance

Taking Exp. wrt $d(n)$ & $\bar{u}(n)$ on eqn. 2, we get at steady state ($\lim_{n\to\infty}$)

$$\lim_{n\to\infty} E\left[\underbrace{\|\tilde{w}(n)\|^2}_{\to Tr(C)}\right] + E\left[\bar{\mu}(n)\,|e_a(n)|^2\right] = E\left[\underbrace{\|\tilde{w}(n-1)\|^2}_{\to Tr(C)}\right] + E\left[\bar{\mu}(n)\,|e_p(n)|^2\right]$$

eliminating using $e_a(n)$

using $e_p(n)$ from eqn. 1a

$$\overbrace{E\left[\bar{\mu}(n)|e_a|^2\right]}^{LHS} = E\left[\bar{\mu}(n)\left|e_a - \mu\, g(e^*(n))\, \|u(n)\|^2\right|^2\right]$$

RHS

RHS

$$= \bar{\mu}(n)\cdot|e_a(n)|^2 + \mu^2\,\bar{\mu}(n)\,\|\bar{u}(n)\|^4\,|g(\cdot)|^2$$

$\|\bar{u}(n)\|^2$

$$- \mu \bar{\mu}(n) \|\bar{u}(n)\|^2 e_a(n) g^*(\cdot) \;-\; \mu \bar{\mu}(n) \|\bar{u}(n)\|^2 e_a^*(n) g(\cdot)$$

Re-applying the $E[\cdot]$

$$= \underbrace{E\left[\bar{\mu}(n)\,|e_a(n)|^2\right]} + \mu^2 E\left[\|\bar{u}(n)\|^2 |g(\cdot)|^2\right] - \mu E\left[e_a(n) g^*(\cdot)\right] - \mu E\left[e_a^*(n) g(\cdot)\right]$$

$$\therefore \quad \boxed{\mu E\left[\|\bar{u}(n)\|^2 |g|^2\right] = 2\,\mathrm{Re}\left\{E\left[e_a^*(n)\, g(e(n))\right]\right\}_{n \to \infty}}$$

No assumption on LRn $\|$ required

4

EMSE of LMS Algorithm

LMS

$g[e(n)] = e(n)$

$$\mu\, E\left[\|\bar{u}(n)\|^2\, |\check{e}(n)|^2\right] = 2\,\mathrm{Re}\left\{E\left[e_a^*(n)\, \check{e}(n)\right]\right\}_{n\to\infty}$$

using ③ $\check{e}(n) = e_a(n) + v(n)$

$$\mu\, E\left(\|\bar{u}(n)\|^2 \cdot |e_a(n) + v(n)|^2\right) = 2\,\mathrm{Re}\left\{E\left[\underbrace{e_a^*(n)(e_a(n)+v(n))}_{|e_a|^2 + \cancel{e_a^* v}}\right]\right\}$$

$$|e_a|^2 + \underbrace{|v(n)|^2}_{\sigma_v^2} + \cancel{e_a^* v} + \cancel{e_a v^*}$$

$R_u = E[\bar{u}\bar{u}^H]$

$2\,\mathrm{Re}\{E(e_a^*(n)\, e(n))\}$ 0

$$\Rightarrow \mu\, E\left[\|u(n)\|^2\, |e_a(n)|^2\right] + \mu \cdot \underbrace{E\left[\|\bar{u}(n)\|^2\right]}_{\mathrm{Tr}[R_u]} \cdot \sigma_v^2 = 2\, \underbrace{E\left[|e_a(n)|^2\right]}_{\zeta_{LMS}}$$

EMSE $\quad \zeta^{LMS} = \frac{\mu}{2}\left\{ E\left[\|u(n)\|^2 |e_a(n)|^2\right] + \sigma_v^2 \cdot Tr[R_u] \right\}_{n\to\infty}$

(#1) For sufficiently small μ, $e_a(n)$ will be small

$$\Rightarrow E\left[|u|^2 \cdot |e_a|^2\right] \lll \sigma_v^2 \, Tr[R_u]$$

$$\boxed{\zeta^{LMS} \simeq \frac{\mu \, \sigma_v^2 \, Tr[R_u]}{2}}$$

$Tr[R_u] \rightarrow M\, r(0)$, $\quad R_u = \begin{bmatrix} r(0) & r(1) & \cdots \\ \vdots & \ddots & \\ & & r(0) \end{bmatrix}_{M\times M}$, $\quad r(0) = E[|u(n)|^2]$

(#2) Separation Principle: Over a wide range of μ, if at the steady state we have $\|\bar{u}(n)\|^2$ independent of $e_a(n)$, then,

$$E\left[\|\bar{u}(n)\|^2 \, |e_a(n)|^2\right] = E\left[\|\bar{u}(n)\|^2\right] E\left[|e_a(n)|^2\right]$$

$$= Tr[R_u] \qquad \xi^{LMS}$$

Then

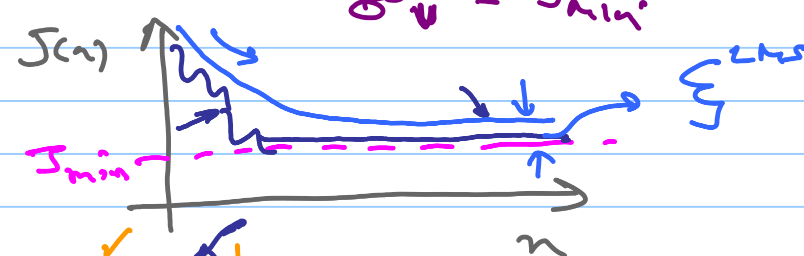
$$\boxed{\xi^{LMS} \approx \frac{\mu \sigma_v^2 \, Tr[R_u]}{2 - \mu \, Tr[R_u]}}$$

, a wide range of μ

EMSE

$$\sigma_v^2 = J_{min} = \sigma_d^2 - \bar{w}_o^H \bar{p}$$

$\sigma_v^2 = J_{min}$



$e(n)\,\bar{u}(n)$

$sgn[e(n)]\,\bar{u}(n)$

$\mu \longrightarrow \mu(n) \downarrow, \; n \to \infty$