EE6110 Aug.-Nov., 2021

Class #8 (Sep. 09, 2021)

Note Title 27-03-2020

not analytic

(*) $\frac{\partial}{\partial \bar{w}}\left(\bar{w}^H \bar{p}\right) = \bar{0}$; $\frac{\partial}{\partial \bar{w}}\left(\bar{p}^H \bar{w}\right) = \bar{p}^H$;

$\frac{\partial}{\partial \bar{w}^*}\left(\bar{w}^H \bar{p}\right) = \bar{p}$; $\frac{\partial}{\partial \bar{w}^*}\left(\bar{p}^H \bar{w}\right) = \bar{0}$;

(*) $\frac{\partial}{\partial \bar{w}}\left(\overbrace{\bar{w}^H_{1\times M}}^{\mathrm{I}}\ \overbrace{R_{M\times M}\ \bar{w}_{M\times 1}}^{\mathrm{II}}\right) \to 0 \cdot \mathrm{II} + \bar{w}^H \cdot R$;

$\frac{d}{dx} a(x) b(x) = \left(\frac{d}{dx} a(x)\right) b(x) + \left(\frac{d}{dx} b(x)\right) a(x)$

$\frac{\partial}{\partial \bar{w}^*}\left(\quad \cdot \quad \right) = R\bar{w}$;

* Eigen Analysis in Haykin $\longrightarrow$ Reading (x: Ch. 1)

Ch. 1 (Haykin).

Given measurements $u(1), u(2) \dots\dots u(M)$

#

$$\bar{u}(n) = \begin{bmatrix} u(n) & u(n-1) & \dots\dots & u(n-M+1) \end{bmatrix}^T_{M\times 1}$$

Correlation Matrix (WSS)

$$R_{M\times M} = E\left[\bar{u}(n)\,\bar{u}^H(n)\right]_{M\times M}$$

$$E\left[u(n-k)\,u^*(n-i)\right] \triangleq r(i-k) \leftarrow \text{first index minus the second index}$$

$$R = \begin{bmatrix} r(0) & r(1) & r(2) & \dots & r(M-1) \\ r(-1) & r(0) & r(1) & \dots & r(M-2) \\ r(-2) & r(-1) & r(0) & \dots & r(M-3) \\ \vdots & & & & \\ r(-M+1) & \dots & & & r(0) \end{bmatrix}$$

$r^*(1)$

$r^*(2)$

$R = R^H$; Hermitian

Toeplitz

$r(k) \rightarrow$ even function

$r^*(k) = r(-k)$

(*) $R \rightarrow$ non-negative definite. almost positive definite

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Aside

Ergodicity & Time Averages

$N = 10{,}000$

$R = E\left[\bar{u}(n)\,\bar{u}^H(n)\right]_{M \times M} \rightarrow$ k^{th} lag $r(k) = E\left[u(n)\,u^*(n-k)\right]$

200

$\bar{u}(n) \rightarrow M \times 1$

$u(1), u(2) \cdots u(N)$

$$\hat{r}(k, N) \triangleq \frac{1}{N} \sum_{n=0}^{N-1} u(n)\,u^*(n-k)$$

$\frac{1}{N-k}$

$0 \leq k \leq N' = N-1$

$N' \leq \frac{N}{10}$

biased

time averaging

* If $E\left[\hat{r}(k, N)\right] = r(k)$ — the estimate ergodic in the mean → unbiased estimator

* $\lim_{N \to \infty} E\left[\left| r - \hat{r}(N) \right|^2 \right] = 0$ ← estimate is ergodic in the MS sense

$\frac{1}{N}$ vs $\frac{1}{N-k}$

$u(1)$ … $u(N)$

$0 \le k \le N \leftarrow N'$

$r(1)$

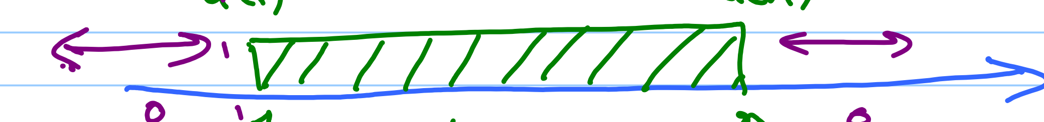
unbiased estimator of $r(\cdot)$

$r(k)$

* auto-covariance formulation
* auto-correlation formulation
* pre-windowed
* post-windowed formation

(*)

0 $u(1)$ $u(n)$



0 1 n 100 0

ergodic

20x20

$r(0), r(1) \ldots\ldots r(k)$

$k > n$ ✗

$k = n$?

$k < n$? ✓

$N-k$

Finite Amount of Data N $\longleftrightarrow$ $R, r(k)$

$k \leftarrow$ lags ?

$\rightarrow \frac{1}{N} \sum \overbrace{u(n)\, u^*(n-k)}^{k}$

$\underbrace{u(1)^2 \quad u(4)}_{3}$

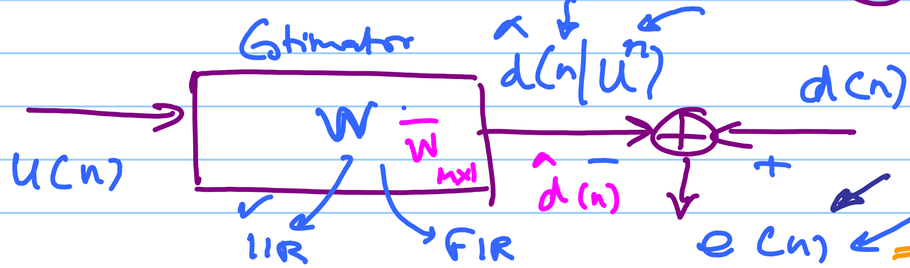
$N = 15$

1 → 15
1 → 14
2 → 13
14 → 1

Maximum Entropy Spectral Estimation → MEM

JP Burg

Wiener-Hopf Equations



$e(n) = d(n) - \hat{d}(n|\mathcal{U}^n)$

$$\bar{u}(n)_{M\times 1} = \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(n-M+1) \end{bmatrix};$$

$\mathcal{U}^n = (u(1), u(2) \ldots u(n+1))$

$(u(n) \ldots$

(*) IIR Wiener Filter $\quad \hat{d}(n|U^n) = \sum_{i=0}^{n} w_i^* u(n-i)$

LMMSE $\min_{\vec{W}} E\left[\underbrace{|e(n)|^2}_{e(n)\,e^*(n)}\right]$

complex

$\frac{d}{dW_k}$

$$E\left[u(n-k)\Big(\underbrace{d^*(n) - \sum_{i=0}^{\infty} w_i u^*(n-i)}_{e^*(n)}\Big)\right] = 0 \quad \forall k = 0,1,2,\dots$$

principle of orthogonality

Wiener-Hopf Equations:

$$\sum_{i=0}^{\infty} w_i \underbrace{E\left[u(n-k)u^*(n-i)\right]}_{\triangleq r(i-k)} = \underbrace{E\left[u(n-k)d^*(n)\right]}_{\triangleq p(-k)}, \quad \forall k = 0,1,2\dots$$

$r(i-k)$ ← a.c.f.

$p(-k)$ ← cross-cor. fn.

Finite order Wiener filter.

$$\hat{d}(n|U^n) = \sum_{i=0}^{M-1} w_i^* u(n-i)$$

$$\hat{d}(n) = \bar{w}^H \bar{u}(n)$$

$\bar{w} \Rightarrow M\times1$

orthogonality condition $E\left[\bar{u}(n)\, e^*(n)\right] = \bar{0}_{M\times 1}$

$(d(n) - \hat{d}(n))^*$

$d^*(n) - \bar{u}^H \bar{w}$

$$\underbrace{E\left[\bar{u}(n)\,\bar{u}^H(n)\right]}_{R_{M\times M}} \bar{w} = \underbrace{E\left[\bar{u}(n)\, d^*(n)\right]}_{\bar{p}_{M\times 1}}$$

$\utilde{R}\, \utilde{\bar{w}} = \utilde{\bar{p}}$

$$\bar{w}_0 \leftarrow \boxed{\bar{w}_{opt} = R^{-1}\bar{p}}$$

FIR Weinerfilter (LMMSE)

$u^1(1), u^1(2) \ldots u^1(n)$

$u^2(1), u^2(2) \ldots u^2(n)$

$u^\infty(1)\; u^\infty(2)\; u^\infty(n)$

Error. -

(*) Performance Surface of the L-MMSE filter.

cost fn. $J(\bar{w}) = E\left[e(n)\, e^*(n)\right]$, where $e(n) = \left(d(n) - \bar{w}^H \bar{u}(n)\right)$

$$J(\bar{w}) = \underbrace{E[|d|^2]}_{\sigma_d^2} - \bar{w}^H \bar{p} - \bar{p}^H \bar{w} + \bar{w}^H R \bar{w}$$ Exercise

Put $\bar{w} = \bar{w}_{opt}$

min. MSE

$$J(\bar{w}_{opt}) = \boxed{J_{min}} = \sigma_d^2 - \bar{p}^H \bar{w}_o$$

$\bar{w}_o$

$$= \sigma_d^2 - \bar{p}^H R^{-1} \bar{p}$$

$$= \sigma_d^2 - \bar{w}_o^H R \bar{w}_o$$

Exercise

Canonical form of the Error Surface.

$$J(\bar{w}) = J(\bar{w}) - \underbrace{p^H R^{-1} p}_{} + p^H R^{-1} p$$

$$= \sigma_d^2 - (\ \cdot\ ) + \underbrace{(w - R^{-1}p)^H R (w - R^{-1}p)}_{}$$

$\tilde{w} \rightarrow \bar{w}_0 = R^{-1} p$

$\underbrace{\quad}_{J_{min}}$ $\qquad (\bar{w} - \bar{w}_0)^H R (\bar{w} - \bar{w}_0)$

$$\boxed{J(\bar{w}) = J_{min} + \underbrace{(\bar{w} - \bar{w}_0)^H R (\bar{w} - \bar{w}_0)}_{\geq 0}}$$

Excess MSE ✓