EE6110 Aug.-Nov., 2021

Class #6 (Sep. 03, 2021)

Note Title 27-03-2020

$$y = g(x) + v \sim f_v$$

Comparision of Different Estimators

Measurement Model

$$\boxed{y(k) = h\,x(k) + n(k)} \quad n(k) \sim \mathcal{N}(0, \sigma_n^2)$$

Linear, Gaussian Model

h: known const.

$x(k)$: parameter to be estimated

$y = hx + n$

$k \to kT_s$

$T_s = T_b$

$x(k) \in \{+1, -1\}$ ($+\sqrt{E}, -\sqrt{E}$)

$\in \left\{\frac{1+j}{\sqrt{2}}, \frac{1-j}{\sqrt{2}}, \frac{-1-j}{\sqrt{2}}, \frac{-1+j}{\sqrt{2}}\right\}$

$p_1 = p_2 = \frac{1}{2}$; $P(+1) = p_1$; $P(-1) = p_2$;

$x(k) \to f_x(x)$

"flat-fading" model (h)

LGM

$y = hx + n + \alpha$

Affine   const

$y = mx + c$

$y' = y - \alpha$

$y' = hx + n$

$\bar{y} = H\bar{x} + \bar{N}$

$$y(k) = h\,x(k) + n(k)$$

$$Y(z) = \underset{T}{h\,X(z)} + N(z)$$

1. Zero forcing Estimator (ZF)

ZF

$$\boxed{\frac{y(k)}{h} = \hat{x}_{ZF}(k)}$$

$$(y-\hat{y})^2$$

2. Least Squares Estimator.

LS

$$\min_{\hat{x}} |y-\hat{y}|^2 \quad \text{where} \quad \hat{y} = h\hat{x}$$

$$\frac{\partial}{\partial \hat{x}}(y-\hat{y})^2 \longrightarrow 2(y-\hat{y})h = 0 \Rightarrow \boxed{\hat{x}_{LS}(k) = \frac{hy}{|h|^2} = \frac{y(k)}{h}}$$

$$hy = \hat{y}h$$

$$h\hat{x}h$$

real case

3. Maximum Likelihood Estimator :

ML

$$\max_{x(k)} f(y(k) \mid x(k))$$

$$n(k)$$

$$y(k) = h\,x(k) + n(k)$$

$$\max_{x(k)} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{1}{2}\frac{\left(\overbrace{y(k) - h\,x(k)}\right)^2}{2\sigma_n^2}} \rightarrow \text{monotonic fn:}$$

$$\min_{x(k)} \left(y(k) - \underbrace{h\,x(k)}_{\hat{y}(n)}\right)^2$$

$$\boxed{\hat{x}_{ML}(k) = \frac{y(k)}{h}}$$

4. Maximum A-posteriori estimate : (?)
MAP

$$\max_{x^v} f(x(k) \mid y(k)) \rightarrow \frac{f(y|x)\, f(x)}{f(y)}$$

$p_1 \rightarrow 1$

$p_2 \rightarrow -1$

$$f(y) = \sum_{i=1}^{2} f(y|x_i) f(x_i)$$

MAP Rule: $\hat{x}_{MAP} = x_i$ iff

$$f(x(k) = x_i \mid y(k)) \geq f(x(k) = x_j \mid y(k)) \quad \forall\, j \neq i$$

Bayes' Rule

$$\frac{f(y(k) \mid x_i)\, p(x_i)}{\cancel{f(y(k))}} \geq \frac{f(y_k \mid x_j)\, p(x_j)}{\cancel{f(y(k))}} \quad \forall j \neq i$$

p_1, p_2

$$\frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{1}{2}\frac{(y - h x_i)^2}{\sigma_n^2}}$$

$$\hat{x}_{MAP} = x_i \text{ iff } \boxed{(y(k) - h x_i)^2 \leq (y(k) - h x_j)^2 + C \quad \forall j \neq i \quad \text{where } C = 2\sigma_n^2 \ln\left(\frac{p_i}{p_j}\right)}$$

5. Min. Mean Square Error (MMSE) Estimator :

$$\hat{x}_{MMSE} = E\left[x(k) \mid y(k)\right]$$

$P(x = 1) = p_1$;

$P(x = -1) = p_2$;

$$= \int x(k)\, f_{x|y}(x)\, dx \qquad \text{where} \qquad f_{x|y} = \frac{f_{y|x} f(x)}{f_y(y)}$$

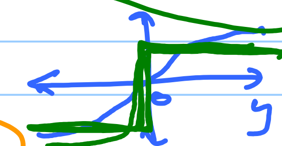
$$\sum_{i=1}^{2} \qquad \cdot\, f_y(y|x_1) f(x_1) + f_y(y|x_2) f(x_2)$$

$$p_1 = p_2 = \frac{1}{2}$$

$$\hat{x}_{MMSE} = \frac{1 \cdot e^{-\frac{(y-h)^2}{2\sigma_n^2}} \cdot p_1 + (-1) \cdot e^{-\frac{(y+h)^2}{2\sigma^2}} \cdot p_2}{e^{-\frac{(y-h)^2}{2\sigma_n^2}} \cdot p_1 + e^{-\frac{(y+h)^2}{2\sigma^2}} \cdot p_2}$$

$$e^{-\frac{(y^2+h^2)}{2\sigma_n^2}}$$

$$\boxed{\hat{x}_{MMSE} = \frac{\sinh(hy/\sigma_n^2)}{\cosh(hy/\sigma_n^2)} = \tanh\left(\frac{hy}{\sigma_n^2}\right)}$$



6. Linear MMSE Estimator. $\min_{W} E\left[\left(x(k) - \hat{x}(k)\right)^2\right]$

where $\hat{x}(k) = W y(k)$