# Solutions of Tutorial-2 (Q.7-Q8)

## Solution -7

(a) The equivalent discrete-time impulse response of the channel is :

$$h(t) = \sum_{n=-1}^{1} h_n \delta(t - nT) = 0.3\delta(t + T) + 0.9\delta(t) + 0.3\delta(t - T)$$

If by $\{c_n\}$ we denote the coefficients of the FIR equalizer, then the equalized signal is :

$$q_m = \sum_{n=-1}^{1} c_n h_{m-n}$$

which in matrix notation is written as :

$$\begin{pmatrix} 0.9 & 0.3 & 0. \\ 0.3 & 0.9 & 0.3 \\ 0. & 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The coefficients of the zero-force equalizer can be found by solving the previous matrix equation. Thus,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{pmatrix}$$

(b) The values of $q_m$ for $m = \pm 2, \pm 3$ are given by

$$\begin{aligned} q_2 &= \sum_{n=-1}^{1} c_n h_{2-n} = c_1 h_1 = -0.1429 \\ q_{-2} &= \sum_{n=-1}^{1} c_n h_{-2-n} = c_{-1} h_{-1} = -0.1429 \\ q_3 &= \sum_{n=-1}^{1} c_n h_{3-n} = 0 \\ q_{-3} &= \sum_{n=-1}^{1} c_n h_{-3-n} = 0 \end{aligned}$$

## Solution-8:

(a) The output of the zero-force equalizer is :

$$q_m = \sum_{n=-1}^{1} c_n x_{m_n}$$

With $q_0 = 1$ and $q_m = 0$ for $m \neq 0$, we obtain the system :

$$\begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solving the previous system in terms of the equalizer's coefficients, we obtain :

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0.000 \\ 0.980 \\ 0.196 \end{pmatrix}$$

(b) The output of the equalizer is :

$$q_m = \begin{cases} 0 & m \leq -4 \\ c_{-1}x_{-2} = 0 & m = -3 \\ c_{-1}x_{-1} + c_0 x_{-2} = -0.49 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_0 x_2 + x_1 c_1 = 0.0098 & m = 2 \\ c_1 x_2 = 0.0098 & m = 3 \\ 0 & m \geq 4 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0, \ldots\}$$

and its span is 6 symbols.

## Solution-9

The optimum tap coefficients of the zero-force equalizer can be found by solving the system:

$$\begin{pmatrix} 1.0 & 0.3 & 0.0 \\ 0.2 & 1.0 & 0.3 \\ 0.0 & 0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.3409 \\ 1.1364 \\ -0.2273 \end{pmatrix}$$

The output of the equalizer is :

$$q_m = \begin{cases} 0 & m \leq -3 \\ c_{-1}x_{-1} = -0.1023 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_1 x_1 = -0.0455 & m = 2 \\ 0 & m \geq 3 \end{cases}$$

Hence, the residual ISI sequence is :

$$\text{residual ISI} = \{\ldots, 0, -0.1023, 0, 0, 0, -0.0455, 0, \ldots\}$$

## Solution-10

(a) If we assume that the signal pulse has duration $T$, then the ouput of the matched filter at the time instant $t = T$ is :

$$\begin{aligned} y(T) &= \int_0^T r(\tau)s(\tau)d\tau \\ &= \int_0^T (s(\tau) + \alpha s(\tau - T) + n(\tau))s(\tau)d\tau \\ &= \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \mathcal{E}_s + n \end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal pulse and $n$ is a zero-mean Gaussian random variable with variance $\sigma_n^2 = \frac{N_0 \mathcal{E}_s}{2}$. Similarly, the output of the matched filter at $t = 2T$ is :

$$\begin{aligned} y(2T) &= \alpha \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \alpha\mathcal{E}_s + n \end{aligned}$$

(b) If the transmitted sequence is :

$$x(t) = \sum_{n=-\infty}^{\infty} I_n s(t - nT)$$

with $I_n$ taking the values $1, -1$ with equal probability, then the output of the demodulator at the time instant $t = kT$ is

$$y_k = I_k \mathcal{E}_s + \alpha I_{k-1} \mathcal{E}_s + n_k$$

The term $\alpha I_{k-1} \mathcal{E}_s$ expresses the ISI due to the signal reflection. If a symbol by symbol detector is employed and the ISI is ignored, then the probability of error is :

$$\begin{aligned}
P(e) &= \frac{1}{2} P(\text{error}|I_n = 1, I_{n-1} = 1) + \frac{1}{2} P(\text{error}|I_n = 1, I_{n-1} = -1) \\
&= \frac{1}{2} P((1+\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2} P((1-\alpha)\mathcal{E}_s + n_k < 0) \\
&= \frac{1}{2} Q\left[\sqrt{\frac{2(1+\alpha)^2 \mathcal{E}_s}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\frac{2(1-\alpha)^2 \mathcal{E}_s}{N_0}}\right]
\end{aligned}$$

(c) To find the error rate performance of the DFE, we assume that the estimation of the parameter $\alpha$ is correct and that the probability of error at each time instant is the same. Since the transmitted symbols are equiprobable, we obtain :

$$\begin{aligned}
P(e) &= P(\text{error at } k | I_k = 1) \\
&= P(\text{error at } k-1) P(\text{error at } k | I_k = 1, \text{error at } k-1) \\
&\quad + P(\text{no error at } k-1) P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\
&= P(e) P(\text{error at } k | I_k = 1, \text{error at } k-1) \\
&\quad + (1 - P(e)) P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\
&= P(e) p + (1 - P(e)) q
\end{aligned}$$

where :

$$\begin{aligned}
p &= P(\text{error at } k | I_k = 1, \text{error at } k-1) \\
&= \frac{1}{2} P(\text{error at } k | I_k = 1, I_{k-1} = 1, \text{error at } k-1) \\
&\quad + \frac{1}{2} P(\text{error at } k | I_k = 1, I_{k-1} = -1, \text{error at } k-1) \\
&= \frac{1}{2} P((1+2\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2} P((1-2\alpha)\mathcal{E}_s + n_k < 0) \\
&= \frac{1}{2} Q\left[\sqrt{\frac{2(1+2\alpha)^2 \mathcal{E}_s}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\frac{2(1-2\alpha)^2 \mathcal{E}_s}{N_0}}\right]
\end{aligned}$$

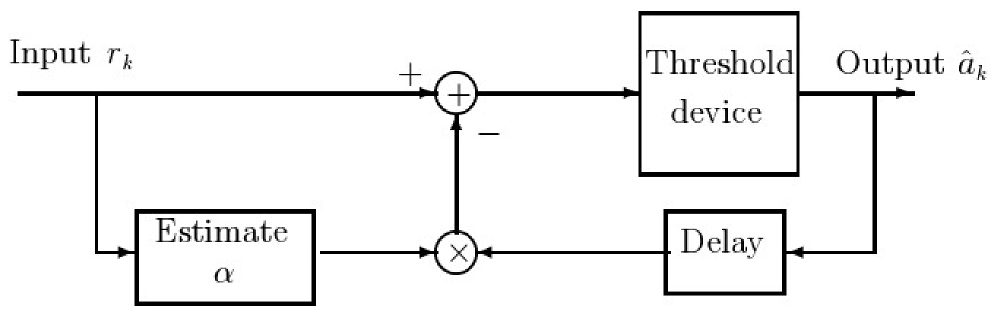
and

$$\begin{aligned} q &= P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\ &= P(\mathcal{E}_s + n_k < 0) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right] \end{aligned}$$

Solving for $P(e)$, we obtain :

$$P(e) = \frac{q}{1-p+q} = \frac{Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}{1 - \frac{1}{2}Q\left[\sqrt{\frac{2(1+2\alpha)^2\mathcal{E}_s}{N_0}}\right] - \frac{1}{2}Q\left[\sqrt{\frac{2(1-2\alpha)^2\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}$$

A sketch of the detector structure is shown in the next figure.




## Solution-11

(a) The spectrum of the band limited equalized pulse is

$$\begin{aligned} X(f) &= \begin{cases} \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(\frac{n}{2W})e^{-j\frac{\pi n f}{W}} & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2W}\left[2 + 2\cos\frac{\pi f}{W}\right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{W}\left[1 + 1\cos\frac{\pi f}{W}\right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $W = \frac{1}{2T_b}$

(b) The following table lists the possible transmitted sequences of length 3 and the corresponding output of the detector.

| | | | |
|---|---|---|---|
| -1 | -1 | -1 | -4 |
| -1 | -1 | 1 | -2 |
| -1 | 1 | -1 | 0 |
| -1 | 1 | 1 | 2 |
| 1 | -1 | -1 | -2 |
| 1 | -1 | 1 | 0 |
| 1 | 1 | -1 | 2 |
| 1 | 1 | 1 | 4 |

As it is observed there are 5 possible output levels $b_m$, with probability $P(b_m = 0) = \frac{1}{4}$, $P(b_m = \pm 2) = \frac{1}{4}$ and $P(b_m = \pm 4) = \frac{1}{8}$.

(c) The transmitting filter $G_T(f)$, the receiving filter $G_R(f)$ and the equalizer $G_E(f)$ satisfy the condition

$$G_T(f)G_R(f)G_E(f) = X(f)$$

The power spectral density of the noise at the output of the equalizer is :

$$\mathcal{S}_\nu(f) = S_n(f)|G_R(f)G_E(f)|^2 = \sigma^2|G_R(f)G_E(f)|^2$$

With

$$G_T(f) = G_R(f) = P(f) = \frac{\pi T_{50}}{2}e^{-\pi T_{50}|f|}$$

the variance of the output noise is :

$$\begin{aligned}
\sigma_\nu^2 &= \sigma^2\int_{-\infty}^{\infty}|G_R(f)G_E(f)|^2df = \sigma^2\int_{-\infty}^{\infty}\left|\frac{X(f)}{G_T(f)}\right|^2 df \\
&= \sigma^2\int_{-W}^{W}\frac{4}{\pi^2T_{50}^2W^2}\frac{|1+\cos\frac{\pi f}{W}|^2}{e^{-2\pi T_{50}|f|}}df \\
&= \frac{8\sigma^2}{\pi^2T_{50}^2W^2}\int_0^W\left(1+\cos\frac{\pi f}{W}\right)^2 e^{2\pi T_{50}f}df
\end{aligned}$$

The value of the previous integral can be found using the formula :

$$\begin{aligned}
&\int e^{ax}\cos^n bx dx \\
&= \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)e^a x\cos^{n-1}bx + n(n-1)b^2\int e^{ax}\cos^{n-2}bx dx\right]
\end{aligned}$$

Thus, we obtain :

$$\begin{aligned}\sigma_\nu^2 &= \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \times \left[ \left(e^{2\pi T_{50} W} - 1\right) \left( \frac{1}{2\pi T_{50}} + \frac{2\pi T_{50} + \pi \frac{1}{W^2 T_{50}}}{4\pi^2 T_{50}^2 + 4\frac{\pi^2}{W^2}} \right) \right. \\ &\left. - \frac{4\pi T_{50}}{4\pi^2 T_{50}^2 + \frac{\pi^2}{W^2}} \left(e^{2\pi T_{50} W} + 1\right) \right]\end{aligned}$$

To find the probability of error using a symbol by symbol detector, we follow the same procedure as in Section 9.3.2. The results are the same with that obtained from a 3-point PAM constellation $(0, \pm 2)$ used with a duobinary signal with output levels having the probability mass function given in part (b). An upper bound of the symbol probability of error is :

$$\begin{aligned}P(e) &< P(|y_m| > 1 | b_m = 0) P(b_m = 0) + 2P(|y_m - 2| > 1 | b_m = 2) P(b_m = 2) \\ &\quad +2P(y_m + 4 > 1 | b_m = -4) P(b_m = -4) \\ &= P(|y_m| > 1 | b_m = 0) \left[ P(b_m = 0) + 2P(b_m = 2) + P(b_m = -4) \right] \\ &= \frac{7}{8} P(|y_m| > 1 | b_m = 0)\end{aligned}$$

But

$$P(|y_m| > 1 | b_m = 0) = \frac{2}{\sqrt{2\pi}\sigma_\nu} \int_1^\infty e^{-x^2/2\sigma_\nu^2} dx$$

Therefore,

$$P(e) < \frac{14}{8} Q\left[\frac{1}{\sigma_\nu}\right]$$

## Solution-12

Since the partial response signal has memory length equal to 2, the corresponding trellis has 4 states which we label as $(I_{n-1}, I_n)$. The following figure shows three frames of the trellis. The labels of the branches indicate the output of the partial response system. As it is observed the free distance between merging paths is 3, whereas the Euclidean distance is equal to

$$d_E = 2^2 + 4^2 + 2^2 = 24$$