EE 504 : Adaptive Signal Processing

Tutorial 1

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Estimation Theory – Bias, CRLB, & Conditional Mean

1. Random variable X ~ $\mathcal{N}(\mu, \sigma^2)$ is measured using samples $\{x_1, x_2, ..., x_N\}$. Consider the following estimator for μ:

$$\hat{\mu}(N) = \frac{1}{N+a}\sum_{i=1}^{N} x_i$$

   where $a \geq 0$. For what value of a is $\hat{\mu}(N)$ an unbiased estimator of μ?

2. Suppose $\{z_1, z_2, ..., z_N\}$ are samples from a Gaussian distribution with unknown mean μ, and unknown variance σ^2. We consider the following sample mean and sample variance estimators (assuming ergodicity), namely:

$$\bar{z} = \frac{1}{N}\sum_{i=1}^{N} z_i \qquad (1.1)$$

   and

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(z_i - \bar{z})^2 \qquad (1.2)$$

   Is $\hat{\sigma}^2$ an unbiased estimator of σ^2? (Hint: Show that $E[\hat{\sigma}^2] = (N-1)\sigma^2 / N$.)

3. Suppose that N independent observations $\{x_1, x_2, ..., x_N\}$ are made of an r.v. x that is Gaussian with pdf

$$f_X(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

   Case1: Only μ is unknown. Derive the Cramer-Rao Lower Bound (CRLB) of $E[\tilde{\mu}^2(N)]$ for $\tilde{\mu} = \mu - \hat{\mu}$ for an unbiased estimate of μ.

   Case2: Only σ is unknown. Find CRLB for $E[(\tilde{\sigma}^2(N))^2]$ for $\tilde{\sigma} = \sigma - \hat{\sigma}$.

   Case3: Both μ and σ are unknown. Here, compute the inverse of the Fischer Information Matrix, i.e.,

$$\mathbf{J}^{-1}(\boldsymbol{\theta}), \text{where } \boldsymbol{\theta} = [\mu \quad \sigma^2] \qquad (1.3)$$

4. If $\mathbf{x}_{M\times1}$ and $\mathbf{y}_{N\times1}$ are individually Gaussian, and are also jointly Gaussian random vectors with $\mathbf{m}_x = E[\mathbf{x}], \mathbf{m}_y = E[\mathbf{y}]$, and $\mathbf{R}_{xy} = E[(\mathbf{x}-\mathbf{m}_x)(\mathbf{y}-\mathbf{m}_y)^H]$, then it can be verified that the MMSE estimate

$$\hat{\mathbf{x}} = E[\mathbf{x} \mid \mathbf{y}] = \mathbf{m}_x + \mathbf{R}_{xy}\mathbf{R}_{yy}^{-1}(\mathbf{y} - \mathbf{m}_y) \qquad (1.4)$$

   and the corresponding MMSE matrix $E[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^H] = \mathbf{R}_{xx} - \mathbf{R}_{xy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}$ (1.5)

   Use this to derive the corresponding expressions when x and y are scalars (r.v.s), with $\rho = E[(x-m_x)(y-m_y)]$.

5. Let X and Y be jointly Gaussian, zero-mean r.v.s and let

$$Z_1 = aY + b; Z_2 = Y^2; Z_3 = \text{Cos}Y; \qquad (1.6)$$

   Then, determine $E(X \mid Z_i), i = 1, 2, \&3$. What relation do they have with $E(X \mid Y)$?

6. Let x, y_1 and y_2 be independent, Gaussian, zero-mean, unit-variance, r.v.s and let $r = \sqrt{y_1^2 + y_2^2}$. Determine the following: (a) $E[x \mid r]$, (b) $E[y_1 \mid r]$.

Linear Algebra -- Basics

7. If for a $M \times M$ matrix $\mathbf{A} = \mathbf{A}^H$, then prove that $\forall \mathbf{x} \in \mathbb{C}^M, \mathbf{x}^H\mathbf{A}\,\mathbf{x}$ is real.

8. If $\mathbf{A}^H$ = -**A,** then show that:
   a. $j\mathbf{A}$ is Hermitian, where $j = \sqrt{-1}$
   b. **A** is unitarily diagonisable, with purely imaginary eigenvalues

9. In the below examples, find the unitary matrix **Q** that diagonalises **A**, i.e., $\mathbf{QAQ}^{\mathrm{H}}=\mathbf{\Lambda}$

$$\begin{bmatrix} 4 & 1-j \\ 1+j & 5 \end{bmatrix}; \quad \begin{bmatrix} 3 & -j \\ j & 3 \end{bmatrix}; \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+j \\ 0 & -1-j & 0 \end{bmatrix}; \quad \begin{bmatrix} 2 & j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 2 & 0 \\ j/\sqrt{2} & 0 & 2 \end{bmatrix}$$

10. If a matrix **A** has eigen-values $\lambda_1 = 0$ for $\mathbf{q}_1 = [1 \quad 2]^T$ and $\lambda_2 = 1$ for $\mathbf{q}_2 = [2 \quad -1]^T$, find the following:
(a) trace(**A**) (b) det(**A**) (c) Can you specify **A**?

11. If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, find $(i)\mathbf{A}^{90} (ii)e^{\mathbf{A}}$

12. Consider a recursion where $\mathbf{x}(0) = [2\ 0\ 2]^T$ is the initial state, and

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) \text{ where } \mathbf{A} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \end{bmatrix} \qquad (1.7)$$

(a) Find the eigen-values of eigen-vectors of **A**.
(b) Find the limiting value $\mathbf{x}(\infty)$, i.e., $\lim_{k\to\infty} \mathbf{x}(k)$.

13. From S. Kaykin "Adaptive Filter Theory" 4th Ed., chapter-1, pp. 89-93, Pbms.# **1, 2, 3, 4, 6***, and the following problems from chapter-2, pp.128-135, **Pbms.# 1, 2, 5, 6, 7, 9, 11, 13, 14, 15*,16,** and **19*.**

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