Tutorial 2

1.

$$\begin{aligned} u(k) &= d(k) - 0.5d(k-1) + 0.25d(k-3) + v(k) & (1) \\ v(k) &= n(k) + 0.25n(k-1) & (2) \end{aligned}$$

The equalizer is of order M. The equalizer coefficients can be derived by solving the Wiener-Hopf equation,

$$\mathbf{R}_u \mathbf{w} = \mathbf{p} \quad (3)$$

where

$$\mathbf{R}_u = \left\{ \begin{bmatrix} u(n) \\ \vdots \\ u(n-M-1) \end{bmatrix} \begin{bmatrix} u(n) & \dots & u(n-M-1) \end{bmatrix} \right\} = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(M-1) \\ r_u(1) & r_u(2) & \dots & r_u(M-2) \\ \vdots & \vdots & \vdots & \vdots \\ r_u(M-1) & r_u(M-2) & \dots & r_u(0) \end{bmatrix} \quad (4)$$

and

$$\mathbf{p} = \left\{ \begin{bmatrix} u(n) \\ \vdots \\ u(n-M-1) \end{bmatrix} d(k-\Delta) \right\} \quad (5)$$

$$r_u(k) = E[u(m)u(m-k)]$$

$$\begin{aligned} r_u(0) &= E[\{d(m) - 0.5d(m-1) + 0.25d(m-3) + v(m)\}\, u(m)] \\ &= E[d^2(m) + 0.25d^2(m-1) + 0.0625d^2(m-3) + v^2(m)] \\ &= 1.3125 + E[v^2(m)] = 1.3125 + 0.3 = 1.6125 & (6) \end{aligned}$$

$$\begin{aligned} r_u(1) &= E[d(m) - 0.5d(m-1) + 0.25d(m-3) + v(m)][d(m-1) - 0.5d(m-2) \\ &+ 0.25d(m-4) + v(m-1)] \\ &= E[-0.5d^2(m-1) + v(m)v(m-1)] = -0.5 + r_v(1) & (7) \\ r_u(2) &= E[d(m) - 0.5d(m-1) + 0.25d(m-3) + v(m)][d(m-2) - 0.5d(m-3) \\ &+ 0.25d(m-5) + v(m-2)] \\ &= E[-0.125d^2(m-3) + v(m)v(m-2)] = -0.125 + r_v(2) & (8) \\ r_u(3) &= E[d(m) - 0.5d(m-1) + 0.25d(m-3) + v(m)][d(m-3) - 0.5d(m-4) \\ &+ 0.25d(m-6) + v(m-3)] \\ &= E[0.25d^2(m-3) + v(m)v(m-3)] = 0.25 + r_v(3) & (9) \\ r_u(4) &= E[d(m) - 0.5d(m-1) + 0.25d(m-3) + v(m)][d(m-4) - 0.5d(m-5) \\ &+ 0.25d(m-7) + v(m-4)] \\ &= E[0.25d^2(m-3) + v(m)v(m-3)] = r_v(4) & (10) \end{aligned}$$

For lags $k \geq 4$, we have $r_u(k) = r_v(k)$.

$$r_v(0) = E[n(k) + 0.8n(k-1)][n(k) + 0.8n(k-1)] = E[n^2(k) + 0.64n^2(k-1)] = 1.64r_n(0) = 0.3 \quad (11)$$

Therefore, $r_n(0) = 0.1829$. Next

$$\begin{aligned} r_v(1) &= E[n(k) + 0.8n(k-1)][n(k-1) + 0.8n(k-2)] = 0.8r_n(0) = 0.1463 &(12)\\ r_v(2) &= E[n(k) + 0.8n(k-1)][n(k-2) + 0.8n(k-3)] = 0 &(13)\\ r_v(3) &= E[n(k) + 0.8n(k-1)][n(k-3) + 0.8n(k-4)] = 0 &(14)\end{aligned}$$

Therefore, $r_v(k) = 0$ for $k \geq 2$. Therefore, we have non-zero auto-correlation for the input signal only upto lag = 3 . They are

$$r_u(0) = 1.1625, \quad r_u(1) = -0.3536, \quad r_u(2) = -0.125, \quad r_u(3) = 0.25 \tag{15}$$

$r_u(k) = 0$ for $k > 3$. For $M = 4$, we have

$$\mathbf{R}_u = \left[\begin{array}{ccc|c} 1.1625 & -0.3536 & -0.125 & 0.25 \\ -0.3536 & 1.1625 & -0.3536 & -0.125 \\ -0.125 & -0.3536 & 1.1625 & -0.3536 \\ \hline 0.25 & -0.125 & -0.3536 & 1.1625 \end{array}\right] \tag{16}$$

$\mathbf{R}_{uu}$ for M = 3 upper left block 3×3 matrix in (16).

Note : Though only the first 4 auto-correlation values are non-zero, the auto-correlation matrix is a non-singular band matrix and the MSE decreases as the order M is increased.
The cross-correlation vector : For M = 3,

$$\mathbf{p} = \left\{ \begin{bmatrix} u(k) \\ u(k-1) \\ u(k-2) \end{bmatrix} d(k-\Delta) \right\} \tag{17}$$

For $\Delta = 0, 1, 2, 3$ respectively, we have

$$\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

For M = 2, $\Delta = 1$

$$\begin{aligned} \mathbf{R}_u &= \begin{bmatrix} 1.1625 & -0.3536 \\ -0.3536 & 1.1625 \end{bmatrix} \\ \mathbf{p} &= \left\{ \begin{bmatrix} d(k) - 0.5d(k-1) + 0.25d(k-3) + v(k) \\ d(k-1) - 0.5d(k-2) + 0.25d(k-4) + v(k-1) \end{bmatrix} d(k-1) \right\} = \begin{bmatrix} -0.5 \\ 1.0 \end{bmatrix} &(19)\\ \mathbf{w} &= \mathbf{R}^{-1}\mathbf{p} = \begin{bmatrix} -0.1856286 \\ 0.8037520 \end{bmatrix} &(20)\end{aligned}$$

---

2. The output of the IIR system is

$$y(k) = 0.9y(k-1) + I(k) \tag{21}$$

The input to the wiener filter is given by

$$u(k) = y(k) + v(k) \tag{22}$$

The WF coefficients is determined by solving the W-H equation $\mathbf{RW} = \mathbf{p}$. For M = 2, we have

$$\begin{bmatrix} r_u(0) & r_u(1) \\ r_u(1) & r_u(0) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \tag{23}$$

$$\begin{aligned} r_u(k) &= r_y(k) + r_v(k), \quad \forall k &(24)\\ p_k &= E[u(m)d(m-k)] = E[u(m)I(m-k)] = E[\{y(m) + v(m)\}\, I(m-k)] \\ &= E[\{0.9y(m-1) + I(m) + v(m)\}\, I(m-k)] &(25)\end{aligned}$$

From (...)

$$\begin{aligned} r_y(0) &= 0.9r_y(1) + 1.0 \\ r_y(1) &= 0.9r_y(0) \end{aligned} \tag{26}$$

Solving (...) gives $r_y(0) = 5.2631$, $r_y(1) = 4.7368$. Likewise $p_0 = 1$ and $p_1 = 0$.

$$\begin{aligned} \begin{bmatrix} r_y(0) + r_v(0) & r_y(1) \\ r_y(1) & r_y(0) + r_v(0) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} &= \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \\ \begin{bmatrix} 5.2631 + 0.4 & 4.7368 \\ 4.7368 & 5.2631 + 0.4 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{27}$$

This gives

$$\mathbf{w} = \begin{bmatrix} 0.5878 & -0.4916 \end{bmatrix} \tag{28}$$

If $\sigma_v^2 = r_v(0) = 0$, we have

$$\mathbf{w} = \begin{bmatrix} 1.0 & 0.9 \end{bmatrix} \tag{29}$$

When there is no additive noise the WF acts as the whitening filter. It whitens the the coloured signal which is the output of the IIR (AR) system.

---

3. minimize $\|\mathbf{w}^T\mathbf{u}\|^2$ subject to $\mathbf{w}^T\mathbf{w} = 1$

The cost functional for constrained optimization is

$$\begin{aligned} J &= \|\mathbf{w}^T\mathbf{u}\|^2 + \lambda^T(\mathbf{w}^T\mathbf{g} - 1) \\ &= \mathbf{w}^T\mathbf{u}\mathbf{u}^T\mathbf{w} + \lambda^T(\mathbf{w}^T\mathbf{g} - 1) \\ &= \mathbf{w}^T\mathbf{R}\mathbf{w} + \lambda(\mathbf{g}^T\mathbf{w} - 1) \end{aligned} \tag{30}$$

to determine whether a maxima or minima is obtained, we must find out if $\mathbf{x}^T\mathbf{R}\mathbf{x}$ should be positive for every nonzero $\mathbf{x}$. $\mathbf{R}$ should be positive definite, therefore, it has a minimizing solution because the eigenvalues of R = 3 and 7. Using the fact that

$$\frac{d(\mathbf{x}^T\mathbf{A}\mathbf{x})}{d\mathbf{x}} = 2\mathbf{x}^T\mathbf{A}\mathbf{x} \tag{31}$$

and

$$\frac{d\mathbf{a}^T\mathbf{x}}{\mathbf{x}} = \mathbf{a}^T \tag{32}$$

$$\frac{dJ}{d\mathbf{w}} = 2\mathbf{w}^T\mathbf{R} + \lambda\mathbf{g}^T = \mathbf{0} \Leftrightarrow 2\mathbf{w}^T\mathbf{R} = -\lambda\mathbf{g}^T \tag{33}$$

$$\frac{dJ}{d\lambda} = 0 \Leftrightarrow \mathbf{g}^T\mathbf{w} = 1 \tag{34}$$

Therefore, we have

$$\mathbf{w} = -\frac{\lambda}{2}\mathbf{R}^{-1}\mathbf{g} \tag{35}$$

From (...) and (...),

$$-\frac{\lambda}{2}\mathbf{g}^T\mathbf{R}^{-1}\mathbf{g} = 1 \tag{36}$$

$$\lambda = -\frac{2}{\mathbf{g}^T\mathbf{R}^{-1}\mathbf{g}} \tag{37}$$

Therefore,

$$\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{g}}{\mathbf{g}^T\mathbf{R}^{-1}\mathbf{g}} \tag{38}$$

(b) minimize $\mathbf{w}^T\mathbf{R}\mathbf{w}$ subject to the constraint $\mathbf{W}^T\mathbf{w} = 1$.
Using the Reyleigh theorem (page ..., Simon Haykin), the solution to

$$\min_{w} \mathbf{w}^T\mathbf{R}\mathbf{w} \tag{39}$$

$\mathbf{w}^T\mathbf{w} = 1$, is the eigen vector of $\mathbf{R}$ corresponding to its smallest eigenvalue, i.e. $\lambda(\mathbf{R}) = 3$.

— —

4.(a) The mathematical expression describing the steepest descent algorithm for Wiener filtering is

$$\mathbf{w}(+1) = (I - \mu\mathbf{R})\mathbf{w}(n) + \mu\mathbf{p} \tag{40}$$

Substituting the values, we obtain

$$\mathbf{w}(n+1) = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} \mathbf{w}(n) + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \tag{41}$$

The wiener optimum $\mathbf{w}_{\text{opt}} = \mathbf{R}^{-1}\mathbf{p}$ is

$$\mathbf{w}_{\text{opt}} = [\ 0.375 \quad -0.125\ ]^T \tag{42}$$

As it is known that the method of steepest descent hits the wiener optimum solution, solving (...) iteratively, we will obtain

$$\mathbf{W} = [\ 0.375 \quad -0.125\ ]^T \tag{43}$$

(b) The value of μ that gives the fastest convergence is given by

$$0 < \mu < \frac{2}{\lambda_{\text{max}}} \tag{44}$$

where λ_{max} is the largest eigenvalue of $\mathbf{R}$. The eigen values of $\mathbf{R}$ are 2,4. Therefore,

$$\mu < 0.5 \tag{45}$$

— —

6.

$$\begin{aligned} \mathbf{w}(n+1) &= \mathbf{w}(n) + \mu\mathbf{u}(n)e^*(n) && (46) \\ e(n) &= d(n) - \mathbf{u}^H(n)\mathbf{w}(n) && (47) \\ d(n) &= \mathbf{w}_0^H\mathbf{u}(n) + e_0(n) && (48) \end{aligned}$$

Substituting (...), (...) in (...), we get

$$\mathbf{w}(n+1) = \left[\mathbf{I} - \mu\mathbf{u}(n)\mathbf{u}^H(n)\right]\mathbf{w}(n) + \mu\mathbf{u}(n)d^*(n) \tag{49}$$

Consider

$$\begin{aligned} \epsilon(n+1) &= \mathbf{w}_o - \mathbf{w}(n+1) = \mathbf{w}_o - \left[\mathbf{I} - \mu\mathbf{u}(n)\mathbf{u}^H(n)\right]\mathbf{w}(n) - \mu\mathbf{u}(n)d^*(n) \\ &= \left[\mathbf{I} - \mu\mathbf{u}(n)\mathbf{u}^H(n)\right]\epsilon(n) - \mu\mathbf{u}(n)e_o^*(n) && (50) \end{aligned}$$

— —

7.

$$\begin{aligned} J(n) &= |e(n)|^2 + \beta\mathbf{w}^H(n)\mathbf{w}(n) \\ &= \sigma_d^2 - \mathbf{w}^H(n)\mathbf{p} - \mathbf{p}^H\mathbf{w}(n) + \mathbf{w}^{-1}\mathbf{R}\mathbf{w}(n) && (51) \end{aligned}$$

Now

$$\frac{\partial J(n)}{\partial \mathbf{w}} = -2\mathbf{p} + 2\mathbf{R}\mathbf{w} + 2\beta\mathbf{w} \tag{52}$$

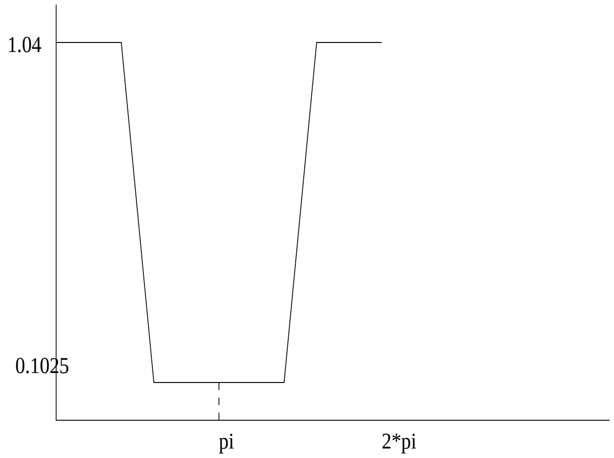
This results in

$$\begin{aligned} \mathbf{w}(n+1) &= \mathbf{w}(n) + \mu(\mathbf{u}(n)d^*(n) - \mathbf{u}\mathbf{u}^H(n)\mathbf{w}(n) - \beta\mathbf{w}(n)) \\ &= \mathbf{w}(n) + \mu[\mathbf{u}(n)(d^*(n) - \mathbf{u}^H(n)\mathbf{w}(n)) - \beta\mathbf{w}(n)] \\ &= \mathbf{w}(n)(1 - \mu\beta) + \mu\mathbf{u}(n)e^*(n) && (53) \end{aligned}$$

8.

$$U(e^{j\omega}) = H(e^{j\omega})I(e^{j\omega}) + V(e^{j\omega}) \tag{54}$$

$$S_{uu}(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_I^2 + \sigma_v^2 \tag{55}$$



$$\frac{\sigma_{\max}(R_{uu})}{\sigma_{\min}(R_{uu})} \leq \frac{S_{uu,\max}(e^{j\omega})}{S_{uu,\min}(e^{j\omega})} = 10.8463 \tag{56}$$