

1. In the figure below, the input $\{I_k\}$ is i.i.d with $E[I_k^2]=1$ and $P[I_k=+1]=1/2= P[I_k=-1]$. The AWGN $v(k)$ has variance $\sigma_v^2=0.2$ and $E[I_k v(i)]=0$ for all k,i .

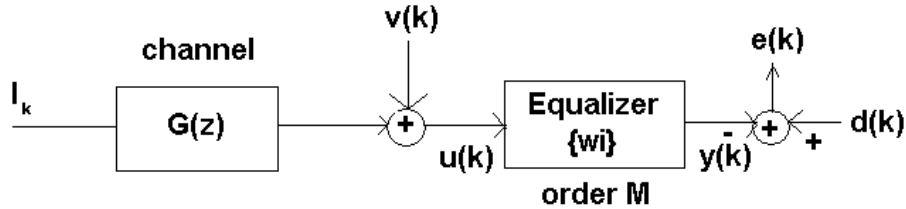


Figure 1 – FIR equalization

The channel $G(z)=1-z^{-1}+0.5 z^{-2}$ and the equalizer has an order equal to M . For the following situations, compute manually \mathbf{R} , \mathbf{p} , and eventually $\mathbf{w}_{opt}=\mathbf{R}^{-1}\mathbf{p}$.

- a) $M=2, d(k)=I(k)$
 - b) $M=2, d(k)=I(k-1)$
 - c) $M=2, d(k)=I(k-2)$
 - d) What is the J_{min} in each of the above cases?
 - e) Now, consider a Decision Feedback Equaliser (DFE) with $M=2$ taps in the feedforward section and 2 taps in the feedback section. For each of the cases in (a), (b), and (c), what will be the resultant \mathbf{R} and \mathbf{p} in this case?
2. A real WSS process $u[n]$ is to be filtered by a 2-tap estimator \mathbf{w} such that $E\{y[n]^2\}$ is minimized where $y[n]=\mathbf{w}^T\mathbf{u}[n]$, subject to $\mathbf{w}^T\mathbf{g}=1$ where $\mathbf{g}=[1 \ -1]^T$.
- a) Given that $r[0]=5$, and $r[1]=2$, use the Lagrange multiplier technique to determine \mathbf{w} . What is the value of the Lagrangian (constant)?
 - b) Repeat part (a) with the alternative constraint $\mathbf{w}^T\mathbf{w}=1$. Specify all values of \mathbf{w} .
3. Steepest Descent Algorithm (SDA) is used to find \mathbf{w}_o with $r[0]=2, r[1]=(1+j)/\sqrt{2}=r^*[-1]$ and $\mathbf{p}=[1+j \ 0]^T$.
- a) Determine the μ that can be used to provide fastest convergence while ensuring stability.
 - b) Starting with $\mathbf{w}=[0 \ 0]^T$ and with μ from above, find $\mathbf{v}^H[25]\mathbf{v}[25]$ where $\mathbf{v}[n] = \mathbf{Q}^H(\mathbf{w}[n]-\mathbf{w}_o)$.
4. Solve problems from “Adaptive filter theory ” by S.Haykin 3rd ed.
- i. pp. 533-535 (Ch 11): 1,2,3,8,9,10.
 - ii. pp. 587-588 (Ch 13): 1,3,4,5,6.
5. A uniform real i.i.d sequence $\{d[k]\}$ with $E\{|d[k]|^2\}=1$ is filtered by $H(z)= 1-0.5z^{-1}+(1/3) z^{-3}$ and the resulting output is corrupted by a coloured noise which is a result of AWGN filtered by $1+0.8z^{-1}$ to give the measurements $\{u[k]\}$ where the variance of the AWGN is 0.25 and this AWGN is uncorrelated with $\{d[k]\}$.

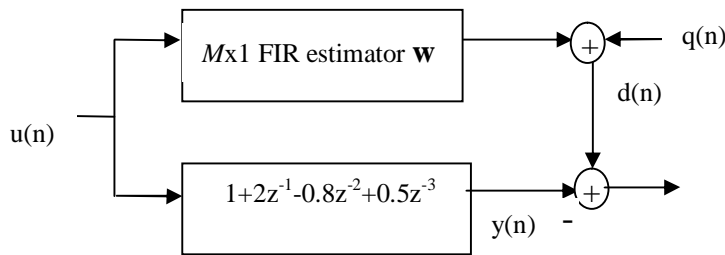
- a) Find \mathbf{R}_{uu} of size 3×3 .
 b) Find a 3×1 $\mathbf{p} = E\{\mathbf{u}[k]d[k-\Delta]\}$ for (i) $\Delta=1$, (ii) $\Delta=4$.

16. In the system identification model shown below, a M^{th} order FIR structure is used to define the estimator \mathbf{w} .

(a) Given that the signal sequence $u(n)$ and the noise sequence $q(n)$ are self and mutually uncorrelated with variance $\sigma_u^2 = 1$, and $\sigma_q^2 = 0.1$, respectively, use the MMSE formulation to

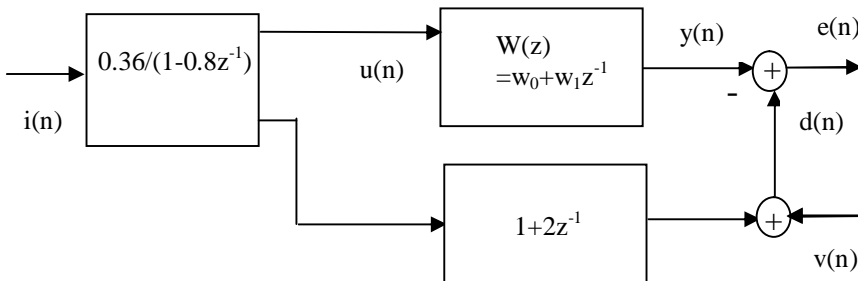
determine \mathbf{w} and J_{\min} for the following choices of M : (i) $M=2$; (ii) $M=3$; (iii) $M=4$; (iv) $M=6$

(b) Based on your above answer, can you make a rough plot of $J_{\min}(M)$ versus M for a suitable range of values for M ?



7. Referring back to Figure 1 in Pbm. 1, the input $\{I_k\}$ is i.i.d with $E[I_k^2]=1$ and $P[I_k=+1]=1/2=P[I_k=-1]$. The AWGN $v(k)$ has variance $\sigma_v^2=1$ and $E[I_k v(i)]=0$ for all k, i , and the channel response $G(z)=1/(1-0.8z^{-1})$. If \mathbf{w} is of order $M=2$, find \mathbf{w}_0 for (a) $\sigma_v^2=1$, (b) $\sigma_v^2=0.3$ and (c) $\sigma_v^2=0$.

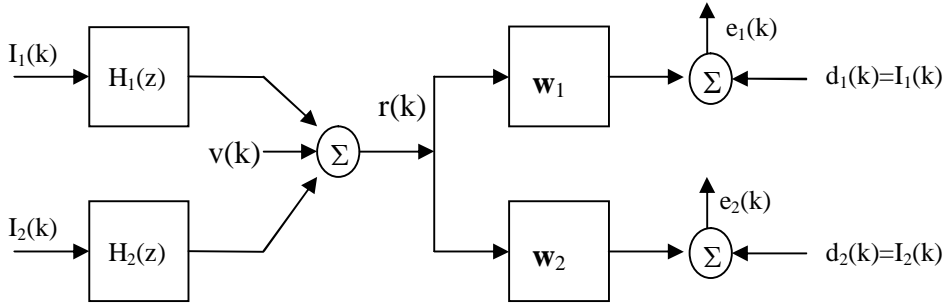
8. Consider the measurement model below, where a linear MSE (Wiener) estimator is to be defined for the equalizer $W(z)$ which will minimize $E[e^2(n)]$. Here, input $i(n)$ is white noise with unit variance, and additive noise $v(n)$ has variance $\sigma_v^2 = 0.10$ and is uncorrelated with $i(n)$.



- (a) Find the correlation matrix \mathbf{R} and the cross-correlation vector \mathbf{p} .
 (b) What is the LMSE estimate for $W(z)$?
 (c) What is the J_{\min} for this LMSE problem?

9. Two mutually uncorrelated i.i.d bipolar bit-streams $I_1(k)$ and $I_2(k)$ with variances σ_1^2 and σ_2^2 are filtered by $H_1(z)=0.8+z^{-1}$ and $H_2(z)=1-0.7z^{-1}$, respectively. Their filtered outputs get added to an AWGN process $v(k)$ with variance σ_v^2 , so as to produce the measurements $r(k)$ as shown in the

figure below. The receiver uses two equalizers, \mathbf{w}_1 and \mathbf{w}_2 , to recover (estimate) $I_1(k)$ and $I_2(k)$, respectively.



(a) If $\sigma_1^2 = 3$, and $\sigma_2^2 = \sigma_v^2 = 0$, specify the MMSE estimate of \mathbf{w}_1 of order $M=5$. *Hint:* At infinite (sufficiently high) SNR, MMSE and zero-forcing estimators are identical.

(b) Now if $\sigma_1^2 = 1$, $\sigma_2^2 = 0.8$, and $\sigma_v^2 = 0.2$, which of the above two equalizers will give a better estimate of the corresponding data symbol? *Hint:* That equalizer providing the lower J_{\min} will be the better one.

10. Given $r(0)=3$ and $r(1)=r(-1)=1$ and $\mathbf{p}=[1 \ 0]^T$ and SDA is used with $\mu=0.25$. Starting with $\mathbf{w}_0=\mathbf{0}$ and order $M=2$ perform SDA iterations (*Hint: cleverly !!*) to find $\mathbf{w}[25]$. Also find the μ which gives fastest convergence.

11. Show that the Newton method can also be written as $\mathbf{w}(n) = \mathbf{w}_o + (1 - 2\mu)^n (\mathbf{w}(0) - \mathbf{w}_o)$. Also, choosing $\mu=1/2$ in this expression ensures 1-step convergence. Where is the catch?

12. Show that another equivalent form for the LMS update equation is given by $\mathbf{w}(n+1) = \mathbf{w}(n) + 2\mu e(n) \mathbf{u}^*(n)$ where error $e(n) = d(n) - \mathbf{w}^T(n) \mathbf{u}(n)$, and "T" is the transpose operation.

13. After defining the weight-error vector by $\boldsymbol{\epsilon}(n) = \mathbf{w}_o - \mathbf{w}(n)$, show that the LMS weight update equation can be re-written as $\boldsymbol{\epsilon}(n) = [\mathbf{I} - \mu \mathbf{u}(n) \mathbf{u}^H(n)] \boldsymbol{\epsilon}(n) - \mu \mathbf{u}(n) e_o^*(n)$ where the estimation error produced by the optimum Wiener filter is given by $e_o(n) = d(n) - \mathbf{w}_o^H \mathbf{u}(n)$.

14. From "Fundamentals of Adaptive Filtering" by A.Sayed (soft-copy), solve the following problems (pp. 332-333):

- (i) Pbm. 6.1 on AR process & (ii) Pbm. 6.3
- (iii) Pbms. 6.4 & 6.5 (Price's Thm.), and (iv) Pbm. 6.6 (Bussgang Thm.).

(v) Consider the update equation for the sign-error LMS algorithm as given below

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \mu \text{sgn}[e^*(n)] \mathbf{u}(n)$$

Using the result of Pbm. 6.4 and 6.5 from above, and assuming $\mathbf{u}(n)$ is Gaussian and $e(n)$ and $v(n)$ (the noise in the data model for $d(n)$) are jointly Gaussian, show that for the complex case, for step-size sufficiently small the excess MSE can be approximated by

$$\zeta^{\text{sign-error LMS}} = \frac{\alpha}{2} (\alpha + \sqrt{\alpha^2 + 4\sigma_v^2}), \text{ where } \alpha = \sqrt{\frac{\pi}{4}} \mu \text{Tr}(\mathbf{R}_u)$$

15. Consider the following 7 measurements $\{u[k]\}$ and the desired signal $\{d[k]\}$

k	1	2	3	4	5	6	7
u[k]	3	3	-3	-3	3	-3	3
d[k]	-2.9	5.1	-1.4	-4.9	1.3	-1	0.9

- Find the least squares estimate $\mathbf{w}_{LS} = [w_0 \ w_1]^T$.
- Starting with $\mathbf{P}(0)=10\mathbf{I}$ and $\mathbf{w}_{RLS}(0)=[0 \ 0]^T$ find $\mathbf{w}_{RLS}(3)$, $\mathbf{P}(3)$, and $e(3)$?

16. Consider again figure 1., used in Pbm.1, where we are interested in formulating the channel and the corresponding equalizer using LS estimators with the following measurements:

k	1	2	3	4	5	6	7
u[k]	1.2	-0.3	0.2	1.5	-0.4	0.1	0.9
I[k]	1	-1	1	1	-1	1	--

Assume $\Delta=0$ and $I(k)=0$ for $n<1$ and $n>6$. The channel has only 2 taps.

- Find its LS estimate using the covariance formulation. Also find LS estimate of equalizer. Observing the convolution of the channel and equalizer, comment on a good choice of the decoding delay Δ .
- Repeat the above using instead the auto-correlation formulation for the LS estimate.