Department of Electrical Engineering

EE-5040 : Adaptive Signal Processing

August 2011 **Tutorial #1** KG / IITM

1. Show that (a) if **A** is invertible, so is $\mathbf{A}^H$
   (b) if **A** is unitary, so is $\mathbf{A}^H$

2. If $\mathbf{A} = \mathbf{A}^H$, then for every $\mathbf{x} \in ¢^M$, $\mathbf{x}^H\mathbf{A}\mathbf{x}$ is real. Prove. (**A** is a M x M matrix.)

3. If $\mathbf{A}^H = -\mathbf{A}$, (a) show that j**A** is Hermitian (j=√-1)
   (b) show that **A** is unitary, diagonalizable and has pure imaginary eigenvalues.

4. Find *k*, *l*, and *m* to make the matrix **A** shown below, a Hermitian matrix

$$A = \begin{bmatrix} -1 & k & -j \\ 3-j5 & 0 & m \\ l & 2+j4 & 2 \end{bmatrix}$$

5. In the below, find the unitary matrix **Q** that diagonalizes **A**, i.e., **QAQ** = **Λ**.

a)

$$A = \begin{bmatrix} 4 & 1-j \\ 1+j & 5 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 4 & 1-j \\ 1+j & 5 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+j \\ 0 & -1-j & 0 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 2 & j/\sqrt{2} & -j/\sqrt{2} \\ -j/\sqrt{2} & 2 & 0 \\ j/\sqrt{2} & 0 & 2 \end{bmatrix}$$

6. If **A** has $\lambda_1 = 0$, and $\lambda_2 = 5$ corresponding to

$$q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad q_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Find a) trace(**A**), b) det(**A**), c) can you specify **A**?

7. Find a real matrix **A** with **A** + α**I** invertible for all real α.

8. Find (i) $\mathbf{A}^{90}$, (ii) $e^{\mathbf{A}}$, if

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

9. Find a 3 x 3 matrix whose rows add up to 1 and show that λ = 1 is an eigenvalue of this matrix. What is the corresponding **q**?

10. Verify if the matrices are unitary; if so, specify their inverses.

a)

$$A = \begin{bmatrix} -j/\sqrt{2} & j/\sqrt{6} & j/\sqrt{3} \\ 0 & -j/\sqrt{6} & j/\sqrt{3} \\ j/\sqrt{2} & j/\sqrt{6} & j/\sqrt{3} \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 3/5 & j4/5 \\ -4/5 & j3/5 \end{bmatrix}$$

c)

$$A = \frac{1}{4}\begin{bmatrix} j+\sqrt{3} & 1-j\sqrt{3} \\ 1+j\sqrt{3} & j-\sqrt{3} \end{bmatrix}$$

d)

$$A = \begin{bmatrix} \frac{1+j}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{j}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-j}{\sqrt{3}} \\ \frac{3+j}{2\sqrt{15}} & \frac{4+3j}{2\sqrt{15}} & \frac{5j}{2\sqrt{15}} \end{bmatrix}$$

11. Consider a recursion where $\mathbf{x}(0) = [\,2\ 0\ 2]^T$ and $\mathbf{x}(k+1) = \mathbf{A}.\mathbf{x}(k)$

$$A = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \end{bmatrix}$$

Find the (i) eigenvalues and the eigenvectors of the matrix **A**, (ii) the value of **x**(k+1), (iii) the limiting value **x**(∞).

12. A WSS process has $\mathbf{R}_{xx}(0) = 1$, and $\mathbf{R}_{xx}(\pm 1) = 0.8$.

a) Choose $\mathbf{R}_{xx}(\pm 2) \geq 0$, such that the process is deterministic like.
b) For your choice of $\mathbf{R}_{xx}(\pm 2)$, obtain an expression for $\mathbf{R}_{xx}(k)$ for all k.

13. State whether the following statements are True. If False, provide the correct statement.
a) an AR process is always asymptotically WSS.
b) an estimator which is ergodic in the mean square error sense need not necessarily be ergodic in the mean.
c) generally speaking, more the more the (extent of) correlation, large will be the eigenvalue spread of the random process.
d) an unitary matrix is diagonalizable only if it has distinct eigenvalues.

14. Consider a r.p. $u(n) = 3e^{j4\pi n} + v(n)$, where v(n) is a Gaussian white noise process with variance $\sigma^2 = 4$.
a) find the 2 x 2 autocorrelation matrix **R.**
b) what is the eigenvalue spread of **R**?
c) find the expression for $\mathbf{R}^6$.

15. We are given random samples $\{x_1, x_2, \ldots, x_N\}$ where each x_i is i.i.d. with $N(\mu,\sigma^2)$. Consider the following estimator for μ,

$$\hat{\mu}(N) = \frac{1}{N+a}\sum_{i=1}^{N} x_i$$

where $a \geq 0$.
(a) For what value(s) of a is the above unbiased estimator of μ? (for small sample case)
(b) For what value(s) of a is the above an <u>asymptotically</u> unbiased estimator of μ?
(c) Prove that the above is a consistent estimator of μ, for all $a \geq 0$.

16. Suppose that N independent observations $\{x_1, x_2, \ldots, x_N\}$ are made of a r.v. X that is Gaussian; i.e.,

$$p(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-(x_i - \mu)^2 / 2\sigma^2]$$

Assuming only μ is unknown, derive the Cramer-Rao lower bound (CRLB) of $E[(\mu - \hat{\mu})^2]$ for an unbiased estimator of μ.

17. For the observations in Pbm.16, now assume that only variance σ^2 is unknown, and derive the CRLB of $E[(\sigma^2 - \hat{\sigma}^2)^2]$ for an unbiased estimator of σ^2.

18. Consider a real 7x1 random vector $\mathbf{u}(n)$ having an auto-correlation **R** with $\lambda_1 > \lambda_2 > \ldots \lambda_7$ and corresponding eigen-vectors $\mathbf{q}_i$, i=1,2,…7. A low-rank model of $\mathbf{u}(n)$, given by $\mathbf{c}(n)$, is constructed using only the eigen-vectors corresponding to the <u>3 largest</u> eigen-values of R, and this $\mathbf{c}(n)$ is transmitted through an AWGN channel.
(a) Describe in matrix-vector notation the operation at the transmitter to obtain $\mathbf{c}(n)$.
(b) The received vector $\mathbf{r}(n) = \mathbf{c}(n) + \mathbf{w}(n)$, where w(n) is AWGN with variance σ_w^2. What is the matrix operation required at the receiver to obtain MMSE estimate $\tilde{\mathbf{u}}(n)$ of $\mathbf{u}(n)$?
(c) If $\lambda_i = 0.9^i$, i=1,2,…7, find the limiting value of σ_w^2 at which the above low-rank model gives the same MSE as the direct transmission of the entire $\mathbf{u}(n)$.