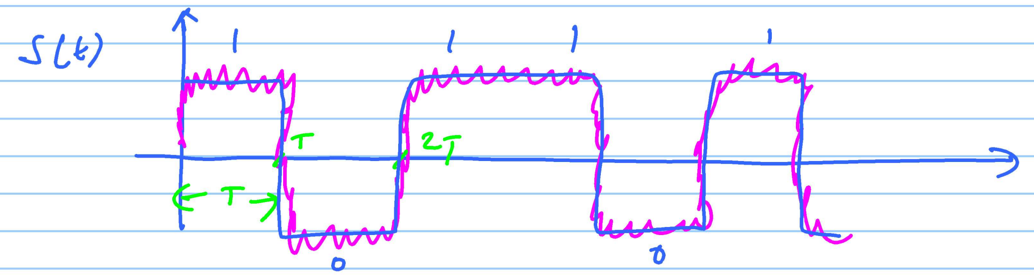
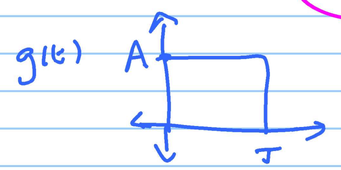
Lesson 2 -- Digital communications thro (band-unlimited) AWGN channels – Signal representation, PAM signals and quick look at timing recovery, PSK and QAM signals, quick look at carrier recovery, orthogonal and multi-dimensional signals, non-linear modulation (FSK and CPM signals)



$$I(k) \longrightarrow \boxed{g(t)} \longrightarrow m(t) = \sum_k I(k)\, g(t-kT) \longrightarrow S(t)$$



lowpass ⟶ bandpass

Signal Representation

Low Pass Signal

Real Signal

Complex signal

Frequency Components (PSD) around "0 Hz"

→ $x(t)$ → Signal

→ $h(t)$ → System

Band Pass Signal

Real Signals

PSD around a carrier f_c ($f_c \neq 0$)

$$\mathrm{Re}\left\{\tilde{s}(t) \cdot e^{j2\pi f_c t}\right\}$$

$$\hat{s}(t) = \frac{1}{\pi t} * s(t)$$

Pre-Envelope

$$s_+(t) = s(t) + j\,\hat{s}(t)$$

$e^{-j2\pi f_c t}$

$e^{+j2\pi f_c t}$

$$\tilde{s}(t) = s_I(t) + j\,s_Q(t)$$

In-phase

Quadrature phase

Complex Envelope

→ Complex Baseband

→ Low-Pass Repn.

A. Pulse Amplitude Modulation (PAM)

$i = 1 \dots M$

$\downarrow$

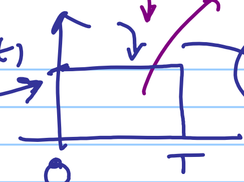
m

$$S_m(t) = Re\left\{ \underbrace{A_m\, g(t)}_{\tilde{S}(t)}\, e^{j2\pi f_c t} \right\}, \quad 0 \le t \le T$$

$$= \boxed{A_m\, g(t) \cos 2\pi f_c t}$$

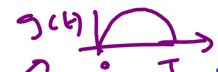
bandpass signal

$kT \le t \le (k+1)T$



$g(t)$ 0 T symbol period

$E_g = \int_0^T g^2(t)\, dt$ Joules



$$A_m = (2m - 1 - M)\, d, \quad m = 1, 2, \dots M$$

$M = 2$

$M = 4$

$m \longrightarrow 1 \quad 2 \quad 3 \quad 4$

$m = 1, 2, 3, 4$

$00 \quad 01 \quad 10 \quad 11$

$n = 4$

$$A_m \in \{ -3d, \underbrace{\ \ }_{2d} -d, \underbrace{\ \ }_{2d} d, \underbrace{\ \ }_{2d} +3d \}$$

Zero-DC signal set

$$E[A_m] = \sum_{m=1}^{4} A_m\, p(A_m) = -3d \times \frac{1}{4} + -d \times \frac{1}{4} + d \times \frac{1}{4} + 3d \times \frac{1}{4} = 0$$

$-5A$

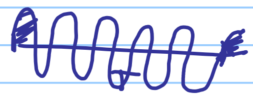
For the m^{th} symbol

symbol energy $E_m = \int_{kT}^{(k+1)T} S_m^2(t)\, dt = \frac{A_m^2}{2} \int_0^T g^2(t)\, dt$

$$S_m(t) = \underbrace{A_m\, g(t) \cos 2\pi f_c t}$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$$



Iff $f_c = \left(P \cdot \frac{1}{T}\right)$

Using the basis-function reprn:

$$\boxed{s_m(t) = s_m\,\phi(t)}$$

Gray coding

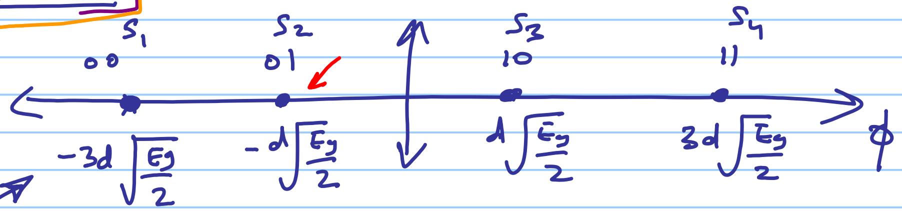
$$\phi(t) = \sqrt{\frac{2}{E_g}}\, g(t) \cos 2\pi f_c t \qquad 0 \le t \le T$$

$$s_m = A_m \sqrt{\frac{E_g}{2}}, \quad m = 1, 2 \ldots M$$

$$(2m-1-M)\,d$$

Check $\int_0^T \phi^2(t)\,dt = 1$

Eg: $\boxed{M = 4}$:



$$d_{min} = 2d\sqrt{\frac{E_g}{2}}$$

minimum Distance $d_{min} = \min_{m,n} \sqrt{(s_m - s_n)^2}$

$$\boxed{d_{min} = d\sqrt{2E_g}}$$

$s_m(t)$ 10 : 01 : 00

s_3 : s_2 : s_1

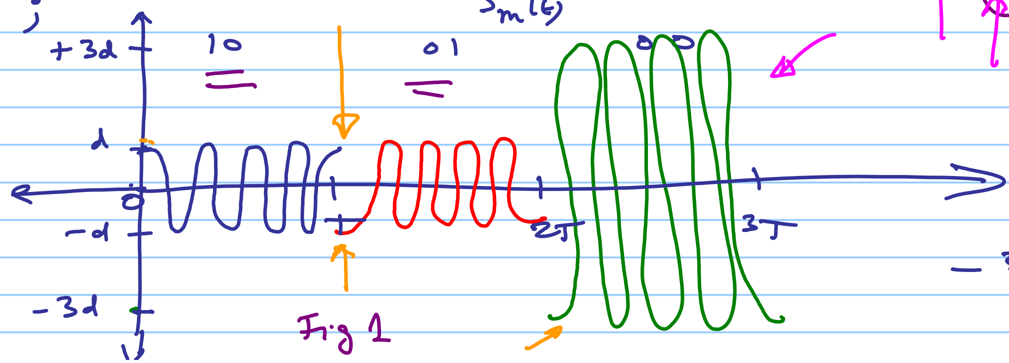
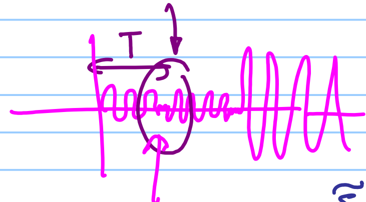
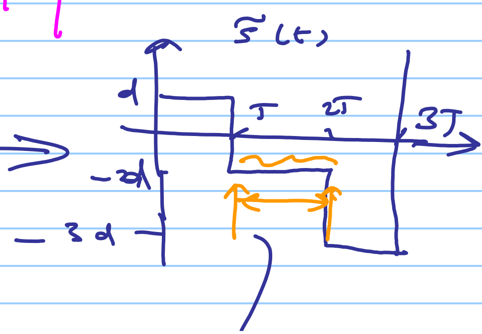
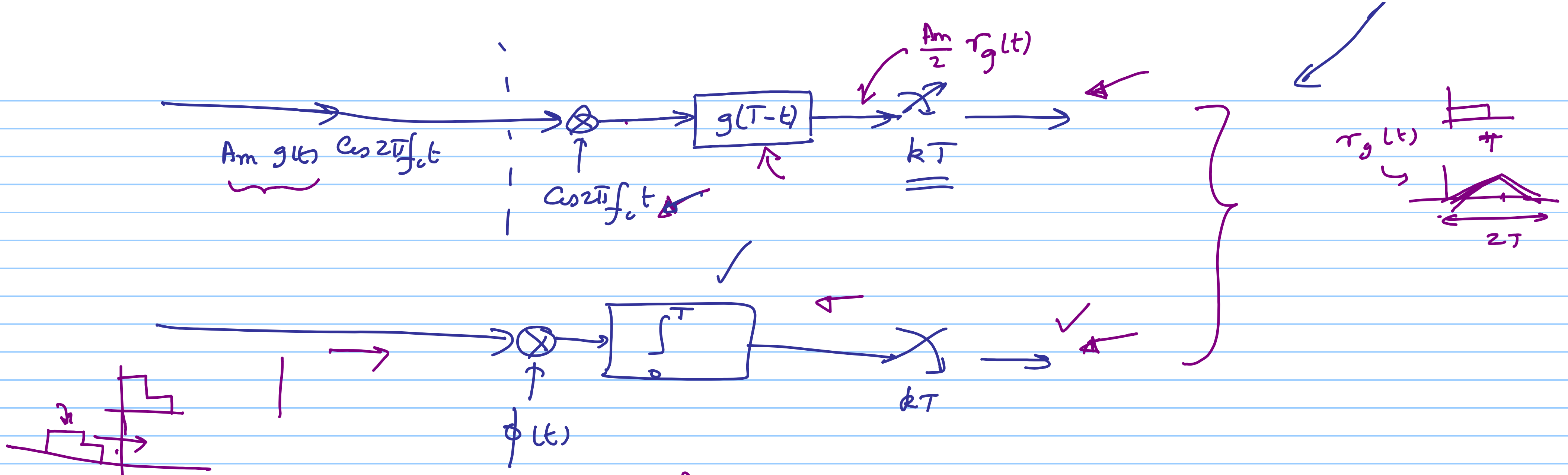


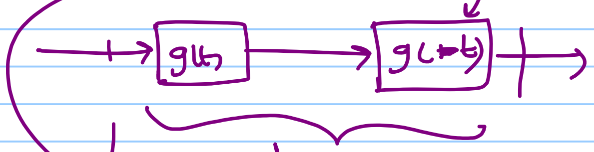
Fig 1



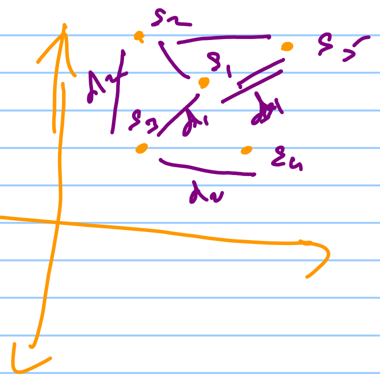




$$s(t) = \int g(\tau)\, g(\tau + t)\, d\tau$$



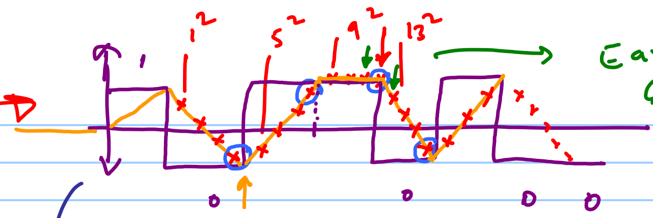
$h(t) =$



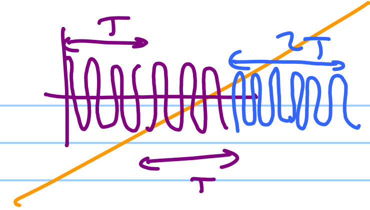
min distance = d_1

Multiplicity = 4

Example

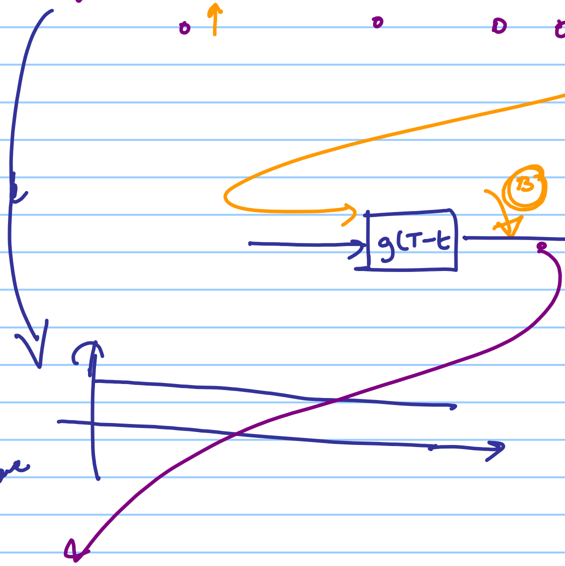


Early-Late Gate ~~Sync.~~ Tracker



$T \rightarrow$ known !!

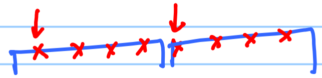
~10 μsec



all "1" sequence

$\rightarrow \boxed{g(T-t)} \rightarrow$ $t = kT_s$ $\rightarrow \boxed{\downarrow 4} \rightarrow$

$T_s = \frac{T}{4}$



1, 5, 9, 13 . . . .

2, 6, 10, 14 - - - -

3, 7, 11, 15 - - - -

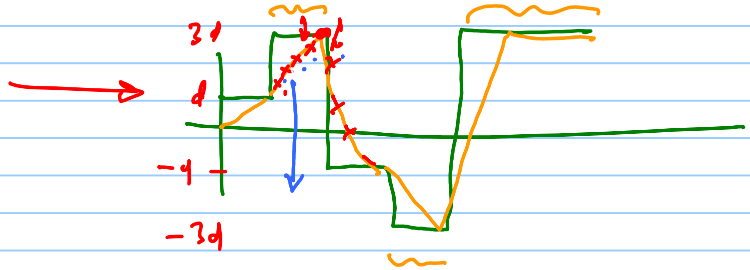
4, 8, 12, 16 - - - -

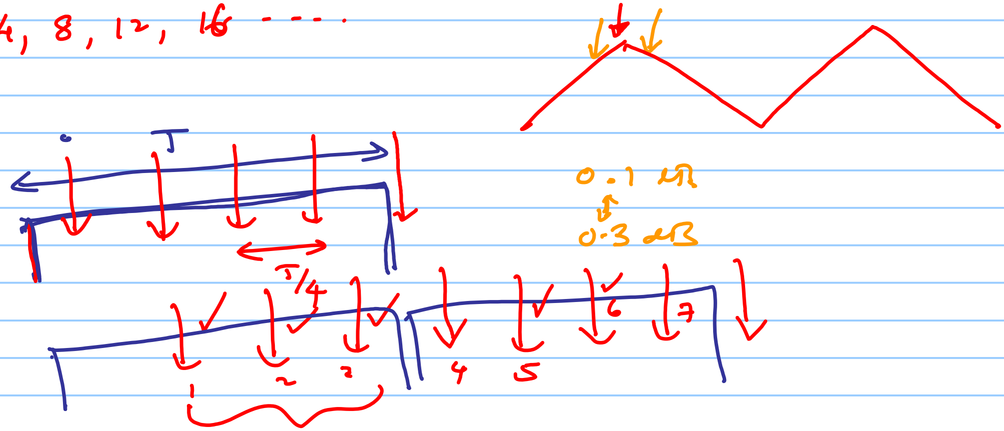
$x^2(k)$ $x^2(k+5)$

$1^2 + 5^2 + 9^2$

$\sum^N$

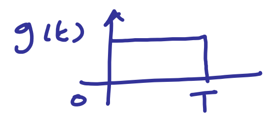
$N \uparrow$





0.1 dB ↕ 0.3 dB

2. Phase Shift Keying (PSK)



$\int_0^T g^2(t)\,dt = E_g$

$$s_m(t) = Re\left\{A\,g(t)\,e^{j\frac{2\pi}{M}(m-1)}\,e^{j2\pi f_c t}\right\}, \quad m = 1, 2 \ldots M, \quad kT < t \le (k+1)T$$

$e^{j\theta_1} e^{j\theta_2} = e^{j(\theta_1+\theta_2)}$

$e^{j\theta} = \cos\theta + j\sin\theta$

$$= A\,g(t)\cos\Big(2\pi f_c t + \underbrace{\frac{2\pi}{M}(m-1)}_{\theta_m}\Big) \quad (1)$$

$\theta_m \in \left\{0, \frac{2\pi}{M}, \frac{4\pi}{M}, \ldots, \frac{(M-1)\cdot 2\pi}{M}\right\}$

$M=4 \quad \theta_m = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$

$$= A\,g(t)\cos\theta_m \cos(2\pi f_c t) - A\,g(t)\sin\theta_m \sin 2\pi f_c t \quad (2)$$

canonical form

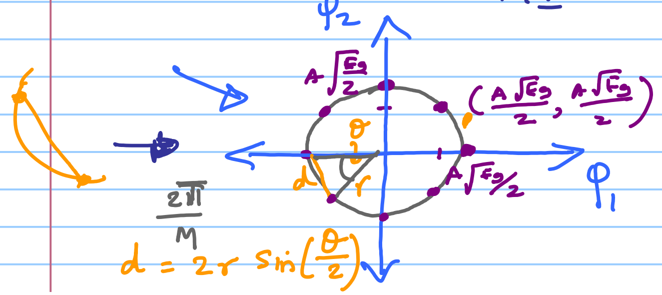
$$= s_{mI}\,\phi_1(t) + s_{mQ}\,\phi_2(t)$$

$f_c = \left(L \cdot \frac{1}{T}\right)$

Exercise: (i) Show that $\phi_1(t) = \sqrt{\frac{2}{E_g}}\,g(t)\cos 2\pi f_c t$; $\phi_2(t) = \sqrt{\frac{2}{E_g}}\,g(t)\sin 2\pi f_c t$;

(ii) Show that $s_{mI} = A\sqrt{\frac{E_g}{2}}\cos\theta_m$; $s_{mQ} = -A\sqrt{\frac{E_g}{2}}\sin\theta_m$;

$I \to$ inphase
$Q \to$ quadrature phase



$M = 8$

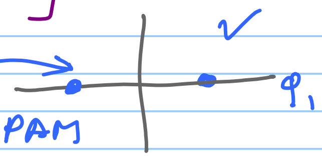
$\theta_m = \left\{0, \frac{\pi}{4}, \pi/2, 3\pi/4, \pi, \ldots\right\}$

$r = A\sqrt{E_g/2}$

8-PSK M-ary PSK

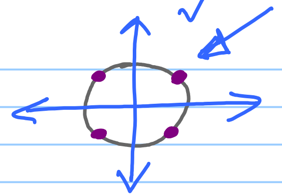
2-ary PSK

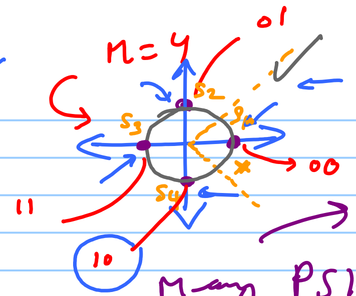
2-ary PAM



QPSK

$M = 4$

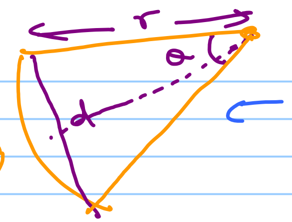


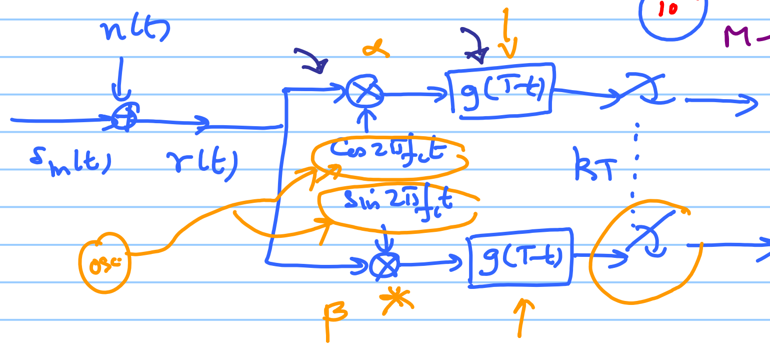


M-ary PSK

Minimum Distance

$$\boxed{d_{min} = 2A\sqrt{\frac{E_g}{2}}\sin\left(\frac{\pi}{M}\right)}$$





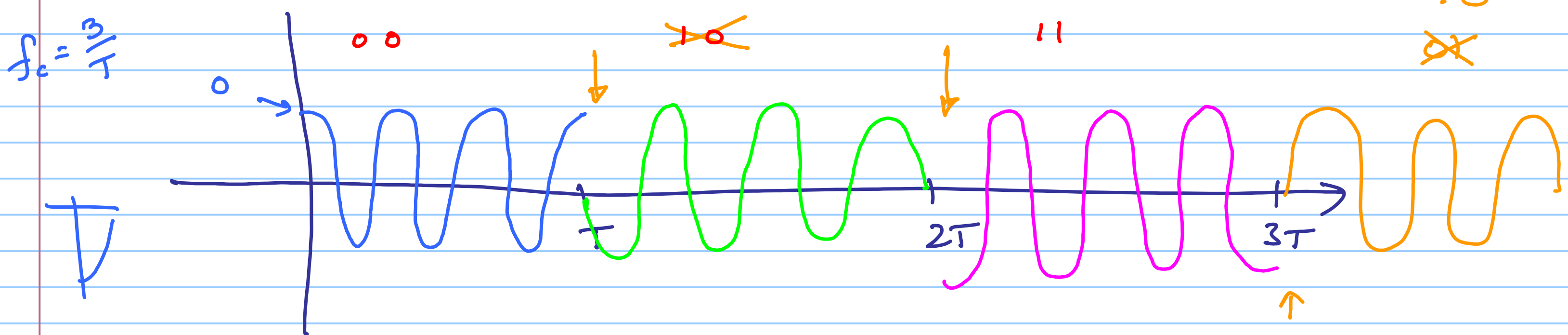
$s_{mI} + n_I$

$s_{mQ} + n_Q$

Feher-QPSK

$M = 4$

$f_c = \frac{3}{T}$

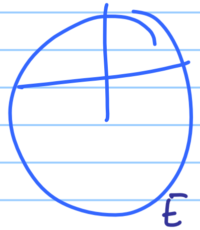


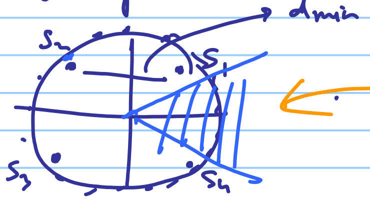
π/4 Offset-QPSK

Staggered-QPSK

~~QPSK~~ 16-PSK

4-ary PSK





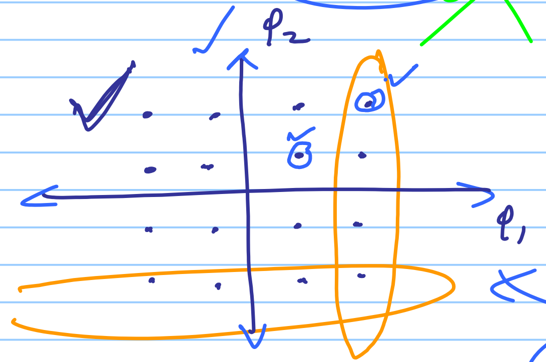
$E_{avg} = |s_1|^2 \cdot p(s_1) + |s_2|^2 p(s_2) + \cdots$

$\cdots\ |s_4|^2 p(s_4)$

$r = \alpha s_m + n$

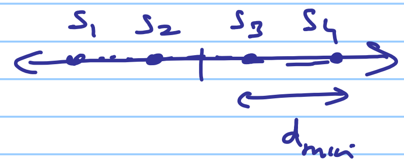
Limiter

16-QAM

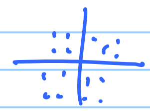


$M_I = M_Q = 4$

~~4-ary PAM~~ 16-ary PAM



E_{avg}



$r = \cancel{\alpha} r_I + j r_Q$

$\alpha r = \alpha r_I + j \alpha r_Q$

$\tan^{-1}\left(\frac{\cancel{\alpha} r_Q}{\cancel{\alpha} r_I}\right)$

3. Quadrature Amplitude Modulation (QAM)

$$s_m(t) = \mathrm{Re}\left\{ \underbrace{\left(A_{m_I} + j A_{m_Q}\right)} g(t)\, e^{j2\pi f_c t} \right\} \qquad kT < t \le (k+1)T$$

$$(\alpha + j\beta)(\gamma + j\delta)$$

$$\alpha\gamma - \beta\delta$$

PAM$_I$ PAM$_Q$

$$= \underbrace{A_{m_I}\, g(t) \cos 2\pi f_c t} - \underbrace{A_{m_Q}\, g(t) \sin 2\pi f_c t}$$

$$= s_{m_I}\, \phi_1(t) + s_{m_Q}\, \phi_2(t)$$

$$\phi_1(t) = \sqrt{\frac{2}{E_g}}\, g(t) \cos 2\pi f_c t$$

$$\& \quad \phi_2(t) = \text{"} \quad \sin 2\pi f_c t$$

1d 00

$b_1\ b_2\ b_3$

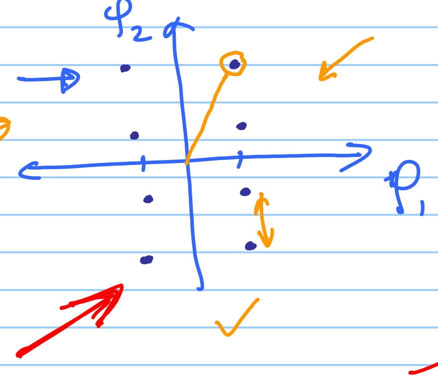
Exercise:

$$\boxed{\begin{aligned} s_{m_I} &= A_{m_I} \sqrt{\frac{E_g}{2}} \\ s_{m_Q} &= A_{m_Q} \sqrt{\frac{E_g}{2}} \end{aligned}}$$

✓ $A_{m_I} = (2m - 1 - M_I)\, d, \quad m = 1, 2 \ldots M_I$ 4

✓ $A_{m_Q} = (2m - 1 - M_Q)\, d, \quad m = 1, 2 \ldots M_Q$ 4

eg: $M_I = 2$; $M_Q = 4$

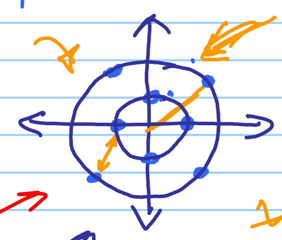


PAM - PSK

eg: $A_{m_I} = (2m - 1 - M_I)d$

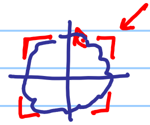
2

4 A_{m_Q} → choose phase



$$\int_{kT}^{(k+1)T} \cancel{\underbrace{\cos 2\pi f_c t \cdot \sin 2\pi f_c t}_{\frac{\sin 4\pi f_c t}{2}} \cdot dt} \to 0$$

$f_c \leftrightarrow T \qquad f_c = \frac{L}{T}$



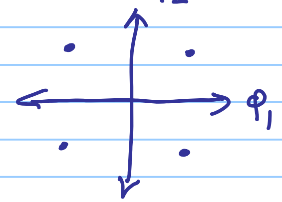
V. fast
V. Big

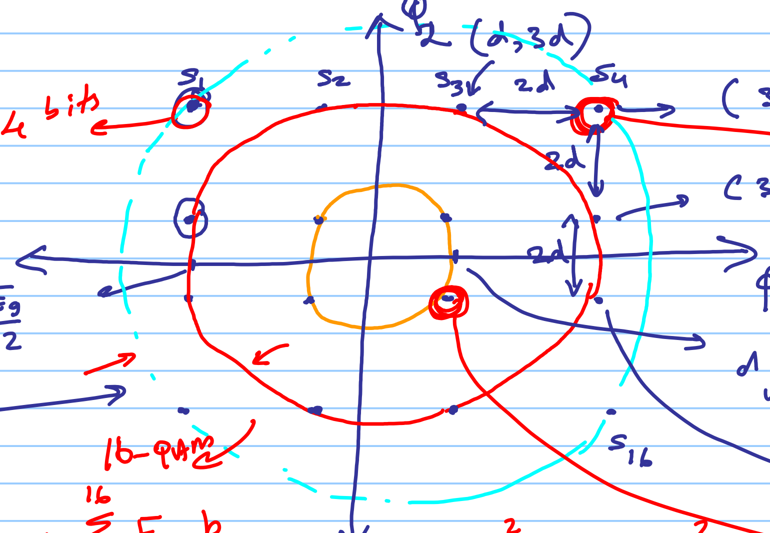
256-QAM
1024-QAM
2048 → QAM
4096 → QAM

"Square" QAM Constellations eg $M_I = 4$; $M_Q = 4$

$M_I = 2$; $M_Q = 2$;

4-QAM





16-QAM

$\left(3d\sqrt{\frac{E_g}{2}}, 3d\sqrt{\frac{E_g}{2}}\right)$

E_{avg}

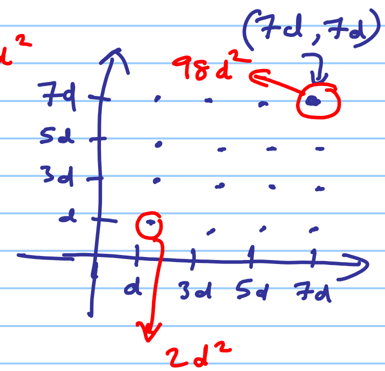
Gray Coding

BICM

Average Energy per symbol $\Big|$ $E_a = \sum_{m=1}^{16} E_m p_m = \frac{4}{16} \times 2d^2 + \frac{8}{16} 10d^2 + \frac{4}{16} 18d^2$ → $2d^2$

$M_I = 8$; $M_Q = 8$

⇒ 64-QAM

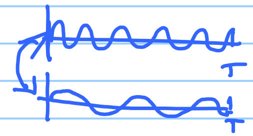


$E_a = 10\, d^2$ Joules

Energy per bit $= E_b$

$E_s = 4 \times E_b$

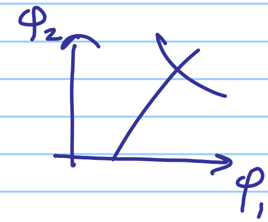
D. orthogonal Signals (FSK)



$$s_m(t) = \mathrm{Re}\left\{ \tilde{s}_m(t)\, e^{j2\pi f_c t} \right\}, \quad m = 1 \cdots M$$

$kT \le t < (k+1)T$

$\boxed{N = M}$



complex BB $\quad \tilde{s}_m(t) = \sqrt{\frac{2E}{T}}\, e^{j2\pi m \Delta f t}$

$f_c \quad f_c + \frac{1}{2T}$

k

i.e., $\quad s_m(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + 2\pi m \Delta f t\right)$

Note that

$$\phi_m(t) = 2 \cos\left(2\pi f_c t + 2\pi m \Delta f t\right)$$

$$\& \quad s_m(t) = \sqrt{\frac{E}{2T}} \cdot \phi_m(t), \quad m = 1 \cdots N$$

cross correlation $\quad \rho_{km} \triangleq \int_0^T \tilde{s}_k(t)\, \tilde{s}_m(t)\, dt$

Exercise: Show that

$$\rho_{km} \triangleq \left(\frac{\sin \pi T (m-k) \Delta f}{\pi T (m-k) \Delta f} \right) \cdot e^{j\pi T (m-k) \Delta f}$$

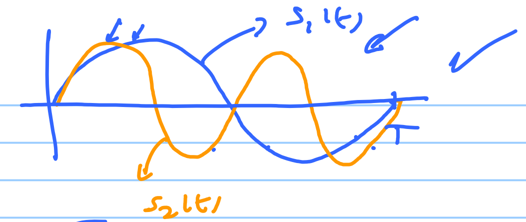
sinc

complex exp.

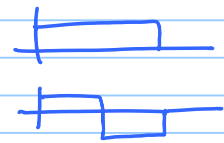
$$\mathrm{Re}\{\rho_{km}\} = \frac{\sin\left(2\pi T (m-k)\Delta f\right)}{2\pi T (m-k)\Delta f}$$

"0" f_c ✓

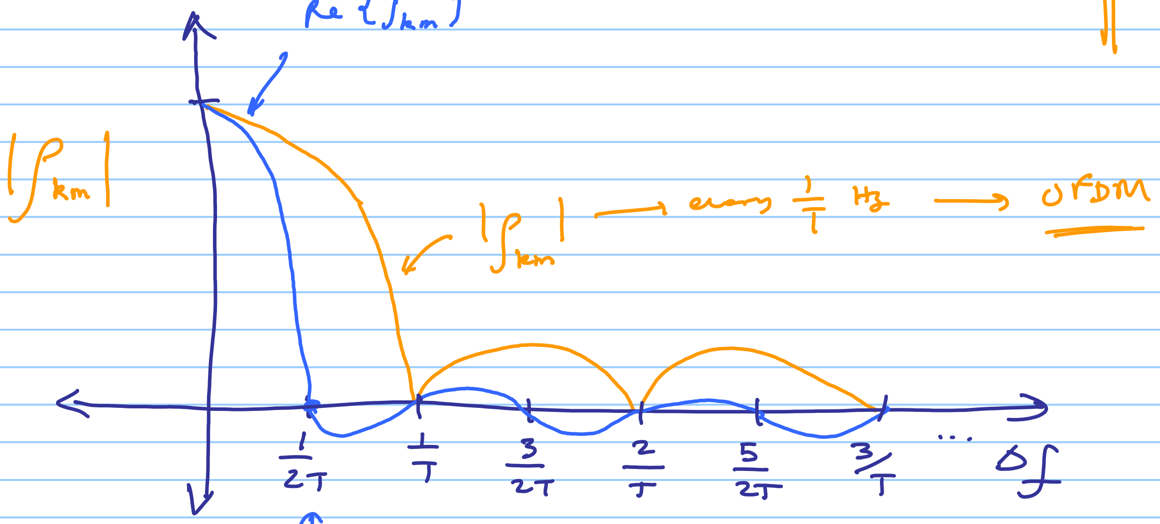
"1" $f_c + \frac{1}{2T}$



$$\int_0^T s_1(t)\, s_2(t)\, dt = 0$$

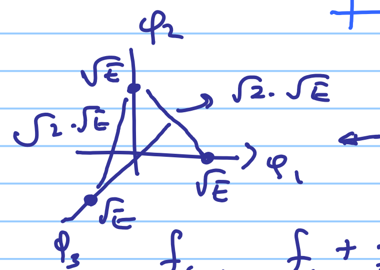


$|\rho_{km}|$



$|\rho_{km}| \longrightarrow$ every $\frac{1}{T}$ Hz $\longrightarrow$ OFDM

$\Delta f = \frac{1}{2T}$



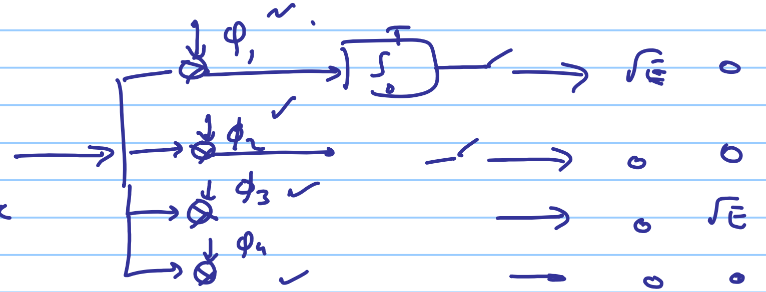
$f_c,\quad f_c + \frac{1}{2T},\quad f_c + \frac{2}{2T},\quad f_c + \frac{3}{2T}$

$\bar{s}_1 = [\sqrt{E}\ \ 0\ \ 0\ \ldots\ 0]$; $\bar{s}_N = [0\ \ 0\ \ldots\ 0\ \sqrt{E}]$;

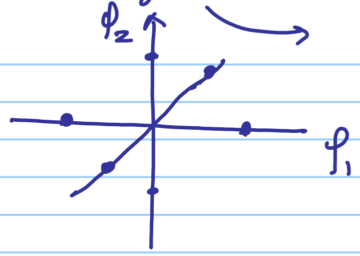
$\bar{s}_2 = [0\ \ \sqrt{E}\ \ 0\ \ldots\ 0]$

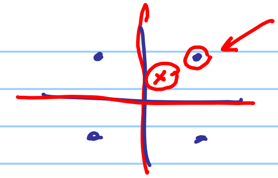
$N \times 1$

$$\bar{s}_m^{\,T}\, \bar{s}_m = E \quad \forall m$$



Bi-orthogonal $\longrightarrow M = \frac{N}{2}$ $\qquad M = N$





Lesson 2C $\qquad$ Optimum Detection Principle

$P(C) \longrightarrow$ MAP rule $\longrightarrow$ ML rule $\xrightarrow[\text{Gaussian}]{\text{Noise}}$ Distance rule

Maximise probability of correct symbol detection

MAP: Maximum A Posteriori Probability Rule

ML: Maximum Likelihood Rule

Distance rule $\downarrow$ Nearest Neighbour