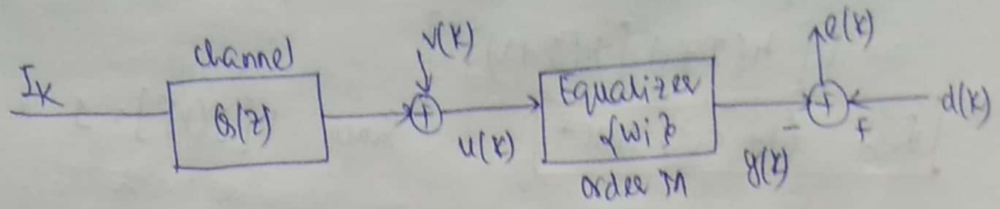
Tutorial 4

Q.1)



Given, $\{I_k\}$ is iid with $E[I_k^2] = 1$

and $P[I_k = +1] = P[I_k = -1] = \frac{1}{2}$. $v(k) \to$ AWGN , $\sigma_v^2 = 0.2$

and $E[I_k v(i)] = 0$ for $\forall$ k, i

$\boxed{G(z) = 1 - z^{-1} + 0.5 z^{-2}}$, equalizer order = M

(a) $M = 2$, $d(k) = I(k)$. $G(z) = 1 - z^{-1} + 0.5 z^{-2}$, $\sigma_v^2 = 0.2$, $E[I(k)^2] =$

$$\therefore u(k) = \begin{bmatrix} 1 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} I(k) \\ I(k-1) \\ I(k-2) \end{bmatrix} + v(k)$$

or, $u(k) = (1 - z^{-1} + 0.5 z^{-2}) I(k) + v(k)$

or, $u(k) = I(k) - I(k-1) + 0.5 I(k-2) + v(k)$

and, $u(k-1) = I(k-1) - I(k-2) + 0.5 I(k-3) + v(k-1)$

Since, $M = 2$

$$\underbrace{\begin{bmatrix} u(k) \\ u(k-1) \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 1 & -1 & 0.5 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} I(k) \\ I(k-1) \\ I(k-2) \\ I(k-3) \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} v(k) \\ v(k-1) \end{bmatrix}}_{V}$$

$$\therefore \boxed{U = HI + V}$$

Now, $R = E[UU^*] = E[(HI + V)(HI + V)^*]$

$= E[HII^*H^* + HIV^* + VI^*H^* + VV^*]$

$= H \underbrace{E[II^*]}_{=1} H^* + 0 + 0 + E[VV^*]$

$\because E[I_k v(i)] = 0$ & $E[I_k^2] = 1$, $\sigma_v^2 = 0.2$

$= HH^* + \sigma_v^2 I_{2\times 2}$

$$= \begin{bmatrix} 2.45 & -1.5 \\ -1.5 & 2.45 \end{bmatrix}$$

Now, $P_1 = E[d(k)U] = E[d(k)(HI + V)]$

$= E[I(k)(HI + V)]$ $(\because d(k) = I(k))$

$= E[I(k)HI + I(k)V]$

$= HE[I(k)I] + 0$

$$= H\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\therefore\ W_{opt_1} = R^{-1}P_1 = \begin{bmatrix}0.6529\\0.3997\end{bmatrix}$$

and, $J_{min_1} = \sigma_I^2 - P_1^H W_{opt_1} = 0.3471$ Ans.

(b) $M=2$, $d(k) = I(k-1)$

$$P_2 = E[d(k)U] = E[I(k-1)(HI+V)] = HE[I(k-1)I] + 0$$

$$= H\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\+1\end{bmatrix}$$

$$\therefore\ W_{opt_2} = R^{-1}P_2 = \begin{bmatrix}-0.2532\\0.2532\end{bmatrix}$$

and, $J_{min_2} = \sigma_I^2 - P_2^H W_{opt_2} = 0.4937$ Ans.

(c) $M=2$, $d(k) = I(k-2)$

$$P_3 = E[d(k)U] = E[I(k-2)(HI+V)] = H\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0.5\\-1\end{bmatrix}$$

$$\therefore\ W_{opt_3} = R^{-1}P_3 = \begin{bmatrix}-0.0733\\-0.4530\end{bmatrix}$$

and, $J_{min_3} = \sigma_I^2 - P_3^H W_{opt_3} = 0.5836$ Ans.

(2) DFE with $M=2$ (feedback taps), $L=2$ (feed forward taps)

(i) $\boxed{\Delta = 0}$

$$a_1 = \begin{bmatrix}U(k)\\U(k-1)\\I(k-1)\\I(k-2)\end{bmatrix} = \underbrace{\begin{bmatrix}1 & -1 & 0.5 & 0\\0 & 1 & -1 & 0.5\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}}_{H_1}\underbrace{\begin{bmatrix}I(k)\\I(k-1)\\I(k-2)\\I(k-3)\end{bmatrix}}_{I_1} + \underbrace{\begin{bmatrix}V(k)\\V(k-1)\\0\\0\end{bmatrix}}_{V_1}$$

$$\therefore\ \boxed{a_1 = H_1 I_1 + V_1}$$

Now,

$$R_1 = E[a_1 a_1^*] = E[(H_1I_1+V_1)(H_1I_1+V_1)^*]$$

$$= H_1 E[I_1 I_1^*] H_1^* + E[V_1V_1^*]$$

$$= H_1 H_1^* + 0.2\, I_{4\times4}$$

$$R_1 = \begin{bmatrix}2.45 & -1.5 & -1 & 0.5\\-1.5 & 2.45 & 1 & -1\\-1 & 1 & 1.2 & 0\\0.5 & -1 & 0 & 1.2\end{bmatrix}$$

and, $$p_1 = E\left[I(k)\, \underline{a}_1\right] = E\left[I(k)(H_1 I_1 + V_1)\right]$$
$$= H_1 E\left[I(k)\, I_1\right]$$
$$= H_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

(ii) $\boxed{\Delta = 1,}$

$$\underline{a}_2 = \begin{bmatrix} V(k) \\ V(k-1) \\ I(k-2) \\ I(k-3) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 1 & -1 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{H_2} \underbrace{\begin{bmatrix} I(k) \\ I(k-1) \\ I(k-2) \\ I(k-3) \end{bmatrix}}_{I_2} + \underbrace{\begin{bmatrix} V(k) \\ V(k-1) \\ 0 \\ 0 \end{bmatrix}}_{V_2}$$

$$\therefore \boxed{\underline{a}_2 = H_2 I_2 + V_2}$$

Now, $$R_2 = E\left[\underline{a}_2\, \underline{a}_2^*\right] = H_2 H_2^* + 0.2\, I_{4\times 4} = \begin{bmatrix} 2.45 & -1.5 & 0.5 & 0 \\ -1.5 & 2.45 & -1 & 0.5 \\ 0.5 & -1 & 1.2 & 0 \\ 0 & 0.5 & 0 & 1.2 \end{bmatrix}$$

and, $$p_2 = E\left[I(k-1)\, \underline{a}_2\right] = H_2 E\left[I(k-1)\, I_2\right] = H_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

or, $$p_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

(iii) $\Delta = 2$

$$\underline{a}_3 = \begin{bmatrix} V(k) \\ V(k-1) \\ I(k-3) \\ I(k-4) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0.5 & 0 & 0 \\ 0 & 1 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{H_3} \underbrace{\begin{bmatrix} I(k) \\ I(k-1) \\ I(k-2) \\ I(k-3) \\ I(k-4) \end{bmatrix}}_{I_3} + \underbrace{\begin{bmatrix} V(k) \\ V(k-1) \\ 0 \\ 0 \end{bmatrix}}_{V_3}$$

$$\therefore \boxed{\underline{a}_3 = H_3 I_3 + V_3}$$

$$\therefore R_3 = E\left[\underline{a}_3\, \underline{a}_3^*\right] = H_3 H_3^* + 0.2\, I_{4\times 4} = \begin{bmatrix} 2.45 & -1.5 & 0 & 0 \\ -1.5 & 2.45 & 0.5 & 0 \\ 0 & 0.5 & 1.2 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix}$$

and, $$p_3 = E\left[I(k-2)\, \underline{a}_3\right] = H_3 E\left[I(k-2)\, I_3\right]$$
$$= H_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

2.

(a)

$I(n) \in \{-1, 1\}$.
$H(z) = 1 - 2z^{-1} + 0.5z^{-2}$
$\sigma_u^2 = 0.3$.
Given $\bar{\boldsymbol{w}}$ is 2-tap, so $\boldsymbol{R_{zz}}$ is a 2x2 matrix and $\bar{\boldsymbol{p}}$ is a 2x1 vector.

Denote $\bar{\boldsymbol{z}} = \begin{bmatrix} z(n) \\ z(n-1) \end{bmatrix}$

$z(n) = I(n) - 2I(n-1) + 0.5I(n-2) + u(n)$
$z(n-1) = I(n-1) - 2I(n-2) + 0.5I(n-3) + u(n-1)$
Here $\sigma_I^2 = \frac{2^2}{12} = 0.33$ as it originates from a uniform distribution.

$\boldsymbol{R_{zz}} = E\left[\bar{\boldsymbol{z}}\bar{\boldsymbol{z}}^T\right] = \begin{bmatrix} (1+4+0.25)\times 0.33 + 0.3 & (-2-1)\times 0.33 + 0.3 \\ (-2-1)\times 0.33 + 0.3 & (1+4+0.25)\times 0.33 + 0.3 \end{bmatrix}$
$\boldsymbol{R_{zz}} = \begin{bmatrix} 2.05 & -0.7 \\ -0.7 & 2.05 \end{bmatrix}$ is the autocorrelation matrix.

(b)

We know, $J_{min} = \sigma_I^2 - \bar{\boldsymbol{w}}_{\boldsymbol{opt}}{}^T \bar{\boldsymbol{p}}_{\boldsymbol{\Delta}}$ for each value of Δ. That is,
$J_{min} = \sigma_I^2 - \bar{\boldsymbol{p}}_{\boldsymbol{\Delta*}}{}^T \boldsymbol{R_{zz}}^{-1} \bar{\boldsymbol{p}}_{\boldsymbol{\Delta*}}$. We only need to calculate the cross correlation vector for different values of Δ to find the optimum Δ^*.

$$\bar{\boldsymbol{p}}_{\boldsymbol{0}} = E\left[I(n)\begin{bmatrix} z(n) \\ z(n-1) \end{bmatrix}\right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sigma_I^2$$
$$\bar{\boldsymbol{p}}_{\boldsymbol{1}} = E\left[I(n-1)\begin{bmatrix} z(n) \\ z(n-1) \end{bmatrix}\right] = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\sigma_I^2$$
$$\bar{\boldsymbol{p}}_{\boldsymbol{2}} = E\left[I(n-2)\begin{bmatrix} z(n) \\ z(n-1) \end{bmatrix}\right] = \begin{bmatrix} 0.5 \\ -2 \end{bmatrix}\sigma_I^2$$
$$\bar{\boldsymbol{p}}_{\boldsymbol{3}} = E\left[I(n-3)\begin{bmatrix} z(n) \\ z(n-1) \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}\sigma_I^2$$

$\boldsymbol{R_{zz}}^{-1} = \begin{bmatrix} 0.5522 & 0.1886 \\ 0.1886 & 0.5522 \end{bmatrix}$
$J_0 = \sigma_I^2 - \bar{\boldsymbol{p}}_{\boldsymbol{0}}{}^T \boldsymbol{R_{zz}}^{-1} \bar{\boldsymbol{p}}_{\boldsymbol{0}} = 0.2720$
$J_1 = 0.1104$
$J_2 = 0.1145$
$J_3 = 0.3180$
It is clear that $J_{min} = 0.1104$, which is attained for $\Delta = 1$.

(c)

$J_{min} = 0.1104$

(d)

The minimum MSE at the optimum Δ is 0.1104. This includes noise contribution as well as that purely due to ISI. The residual noise contribution is given by $\sigma_u^2(\sum w_i^2)$ because variance of filtered white noise is the variance of the unfiltered noise multiplied by the sum of squares of filter's impulse response coefficients.
$\bar{\boldsymbol{w}}_{\boldsymbol{opt}} = \bar{\boldsymbol{p}}_{\boldsymbol{\Delta*}}{}^T \boldsymbol{R_{zz}}^{-1} = [-0.3053, 0.0584]^T$. Thus, noise after Wiener filter is $(0.3053^2 + 0.0584^2) \times 0.3 = 0.0290$.
Thus, the variance of the residual ISI is $J_{min} - \sigma_{u,residual}^2 = 0.1104 - 0.0290 = 0.0814$.
The expression of average probability of error will now be modified as

$$Q\left(\frac{d}{\sqrt{J_{min}}}\right) = Q\left(\frac{d}{\sqrt{\frac{N0}{2}}}\sqrt{\frac{N_0/2}{J_{min}}}\right) = q\left(d\sqrt{\frac{N_0/2}{J_{min}}}\right)$$

Q.3) Given, a uniform real iid sequence $\{d[k]\}$ with $E\{|d[k]|^2\}=1$
At receiver,

$$u[k] = \left(1-0.5z^{-1}+\frac{1}{3}z^{-3}\right)d[k] + (1+0.8z^{-1})v(k)$$

where, $v(k)$: AWGN with variance $= 0.25$

$$\therefore\ u[k] = \begin{bmatrix}1 & -0.5 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}d(k)\\ d(k-1)\\ d(k-2)\end{bmatrix} + \begin{bmatrix}1 & 0.8\end{bmatrix}\begin{bmatrix}v(k)\\ v(k-1)\end{bmatrix}$$

(a). Given, R_{uu} of size 3×3

$$\therefore\ \underbrace{\begin{bmatrix}u(k)\\ u(k-1)\\ u(k-2)\end{bmatrix}}_{U} = \underbrace{\begin{bmatrix}1 & -0.5 & \frac{1}{3} & 0 & 0\\ 0 & 1 & -0.5 & 1/3 & 0\\ 0 & 0 & 1 & -0.5 & 1/3\end{bmatrix}}_{H}\underbrace{\begin{bmatrix}d(k)\\ d(k-1)\\ d(k-2)\\ d(k-3)\\ d(k-4)\end{bmatrix}}_{d} + \underbrace{\begin{bmatrix}1 & 0.8 & 0 & 0\\ 0 & 1 & 0.8 & 0\\ 0 & 0 & 1 & 0.8\end{bmatrix}}_{H'}\underbrace{\begin{bmatrix}v(k)\\ v(k-1)\\ v(k-2)\\ v(k-3)\end{bmatrix}}_{V}$$

$$U = Hd + H'V$$

$$R_{UU} = E[UU^*] = E\left[(Hd+H'V)(Hd+H'V)^*\right]$$
$$= HE[dd^*]H^* + H'E[VV^*]H'^*$$
$$= HH^* + 0.25\,H'H'^*$$
$$= \begin{bmatrix}1.77 & -0.46 & 0.33\\ -0.46 & 1.77 & -0.46\\ 0.33 & -0.46 & 1.77\end{bmatrix}$$ Ans.

(b). $P = E[U(k)\,d(k-\Delta)]$

(i) $\Delta = 1$ $\quad P = E[d(k-1)(Hd+H'V)]$

or, $P = HE[d(k-1)\,d] + 0$

or, $P = H\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}-0.5\\1\\0\end{bmatrix}$ Ans

(ii) $\Delta = 4$ $\quad P = E[d(k-4)(Hd+H'V)]$

$= HE[d(k-4)\,d] + 0$

$= H\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1/3\end{bmatrix}$ Ans

Scanned with CamScanner

4)

Given that

$$U(z) = G(z)\,I(Z) + V(z) \implies (1-0.8z^{-1})\,U(z) = I(z) + (1-0.8z^{-1})\,V(z)$$

$$\implies u(k) = I(k) + v(k) - 0.8\,v(k-1) + 0.8\,u(k-1)$$

Let $z(k) = I(k) + v(k) - 0.8\,v(k-1) \implies u(k) = z(k) + 0.8\,u(k-1)$. Expanding $u(k-1)$,

$$u(k) = z(k) + 0.8z(k-1) + (0.8)^2 u(k-2)$$

Through recursive expansion, we can get

$$u(k) = \sum_{p=0}^{\infty} (0.8)^p\, z(k-p)$$

$\boldsymbol{u} \triangleq \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix}$. Then, $\boldsymbol{R} = \mathbb{E}\left[\boldsymbol{u}\boldsymbol{u}^H\right] = \begin{pmatrix} \mathbb{E}\left[u^2(k)\right] & \mathbb{E}\left[u(k)u(k-1)\right] \\ \mathbb{E}\left[u(k)u(k-1)\right] & \mathbb{E}\left[u^2(k-1)\right] \end{pmatrix}$. For lag l,

$$\begin{aligned}\mathbb{E}\left[u(k)u(k-l)\right] &= \mathbb{E}\left[\left\{\sum_{p=0}^{\infty}(0.8)^p\, z(k-p)\right\}\left\{\sum_{q=0}^{\infty}(0.8)^q\, z(k-l-q)\right\}\right] \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(0.8)^{p+q}\ \mathbb{E}\left[z(k-p)z(k-l-q)\right]\end{aligned}$$

Given that $\mathbb{E}\left[I(k)v(i)\right] = 0\ \forall k, i,\ \mathbb{E}\left[I(k)I(k-m)\right] = \delta(m)$ and $\mathbb{E}\left[v(k)v(k-m)\right] = \delta(m)$. Hence,

$$\begin{aligned}\mathbb{E}\left[z(k-p)z(k-l-q)\right] = {} & \mathbb{E}\left[I(k-p)I(k-l-q)\right] + \mathbb{E}\left[v(k-p)I(k-l-q)\right] \\ & - 0.8\ \mathbb{E}\left[v(k-p-1)I(k-l-q)\right] + \mathbb{E}\left[I(k-p)v(k-l-q)\right] \\ & + \mathbb{E}\left[v(k-p)v(k-l-q)\right] - 0.8\ \mathbb{E}\left[v(k-p-1)v(k-l-q)\right] \\ & - 0.8\ \mathbb{E}\left[I(k-p)v(k-l-q-1)\right] - 0.8\ \mathbb{E}\left[v(k-p)v(k-l-q-1)\right] \\ & + 0.64\ \mathbb{E}\left[v(k-p-1)v(k-l-q-1)\right]\end{aligned}$$

$$\begin{aligned}= {} & \delta(q-p+l) + 0 - 0 + 0 + \sigma_v^2\,\delta(q-p+l) - 0.8\,\sigma_v^2\,\delta(q-p+l-1) - 0 \\ & - 0.8\,\sigma_v^2\,\delta(q-p+l+1) + 0.64\,\sigma_v^2\,\delta(q-p+l)\end{aligned}$$

$$= (1+1.64\,\sigma_v^2)\,\delta(q-p+l) - 0.8\,\sigma_v^2\,\delta(q-p+l-1) - 0.8\,\sigma_v^2\,\delta(q-p+l+1)$$

Thus,

$$\begin{aligned}\mathbb{E}\left[u(k)u(k-l)\right] = (1+1.64\,\sigma_v^2)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(0.8)^{p+q}\ \delta(q-p+l) \\ -\,0.8\,\sigma_v^2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(0.8)^{p+q}\ \delta(q-p+l-1) \\ -\,0.8\,\sigma_v^2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(0.8)^{p+q}\ \delta(q-p+l+1)\end{aligned}$$

But,

$$\begin{aligned}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(0.8)^{p+q}\,\delta(q-p+m) &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(0.8)^{2p-m}\,\delta(q-p+m) && \left[\because f(k)\delta(k-j)=f(j)\delta(k-j)\right] \\ &= \sum_{p=0}^{\infty}(0.8)^{2p-m}\sum_{q=0}^{\infty}\delta(p-m-q) && \left[\because \delta(m)=\delta(-m)\right] \\ &= \sum_{p=0}^{\infty}(0.8)^{2p-m}\ \mathcal{U}(p-m) && \left[\because \sum_{j=0}^{\infty}\delta(k-j)=\mathcal{U}(k),\ \text{unit-step function}\right] \\ &= \begin{cases}\sum_{p=0}^{\infty}(0.8)^{2p-m}, & m<0 \\ \sum_{p=m}^{\infty}(0.8)^{2p-m}, & m\geq 0\end{cases} \quad = \begin{cases}\dfrac{(0.8)^{-m}}{1-(0.8)^2}, & m<0 \\ \dfrac{(0.8)^{m}}{1-(0.8)^2}, & m\geq 0\end{cases} \\ &= \frac{(0.8)^{|m|}}{1-(0.8)^2} \ = \ \frac{(0.8)^{|m|}}{0.36}\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\left[u(k)u(k-l)\right] &= \frac{1}{0.36}\left[(1+1.64\,\sigma_v^2)\,(0.8)^{|l|} - 0.8\,\sigma_v^2\,(0.8)^{|l-1|} - 0.8\,\sigma_v^2\,(0.8)^{|l+1|}\right] \\ &= \frac{1}{0.36}\left\{(0.8)^{|l|} + \sigma_v^2\left[1.64(0.8)^{|l|} - (0.8)^{|l-1|+1} - (0.8)^{|l+1|+1}\right]\right\}\end{aligned}$$

Hence,

$$\mathbb{E}\left[u^2(k)\right] = \frac{1}{0.36}\left[1+0.36\,\sigma_v^2\right] = \frac{25}{9}+\sigma_v^2 \qquad \text{and} \qquad \mathbb{E}\left[u(k)u(k-1)\right] = \frac{0.8}{0.36} = \frac{20}{9}$$

$$\Longrightarrow \boldsymbol{R} = \frac{1}{9}\begin{pmatrix} 25+9\,\sigma_v^2 & 20 \\ 20 & 25+9\,\sigma_v^2\end{pmatrix}$$

Also, $\boldsymbol{p} = \mathbb{E}\left[I(k-\Delta)\boldsymbol{u}\right] = \begin{pmatrix} \mathbb{E}\left[I(k-\Delta)u(k)\right] \\ \\ \mathbb{E}\left[I(k-\Delta)u(k-1)\right] \end{pmatrix}$

$$\mathbb{E}\left[I(k-\Delta)u(k-l)\right] = \mathbb{E}\left[I(k-\Delta)\left\{\sum_{p=0}^{\infty}(0.8)^p\ \left[I(k-l-p)+v(k-l-p)-0.8\,v(k-l-p-1)\right]\right\}\right]$$

$$= \sum_{p=0}^{\infty}(0.8)^p\,\mathbb{E}\left[I(k-\Delta)I(k-l-p)\right]+0+0$$

$$= \sum_{p=0}^{\infty}(0.8)^p\ \delta(p-\Delta+l)\ =\ (0.8)^{\Delta-l}\,\mathcal{U}(\Delta-l)$$

Let $\Delta = 0$. Hence, $\boldsymbol{p} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}$. Also, $\boldsymbol{R}^{-1} = \dfrac{9}{(25+9\,\sigma_v^2)^2-400}\begin{pmatrix} 25+9\,\sigma_v^2 & -20 \\ \\ -20 & 25+9\,\sigma_v^2 \end{pmatrix}$

$$\implies \boldsymbol{w}_0 = \boldsymbol{R}^{-1}\boldsymbol{p} = \frac{9}{(25+9\,\sigma_v^2)^2-400}\begin{pmatrix} 25+9\,\sigma_v^2 \\ \\ -20 \end{pmatrix}$$

(a) $\sigma_v^2 = 1, \quad \implies \boldsymbol{w}_0 = \begin{pmatrix} 0.4048 \\ \\ -0.2381 \end{pmatrix}$

(b) $\sigma_v^2 = 0.3, \implies \boldsymbol{w}_0 = \begin{pmatrix} 0.6788 \\ \\ -0.4901 \end{pmatrix}$

(c) $\sigma_v^2 = 0, \quad \implies \boldsymbol{w}_0 = \begin{pmatrix} 1 \\ \\ -0.8 \end{pmatrix}$

Note:

It can be observed that as $SNR \to \infty$, $(i.e.,\ \sigma_v^2 \to 0)$, $\boldsymbol{w}_0$ approaches $[1\ \ -0.8]^T$. The corresponding transfer function $W(z) = 1 - 0.8z^{-1}$ is the inverse of channel function $(i.e.,\ W(z) = G^{-1}(z))$. Thus, LMSE filter completely cancels the Inter-Symbol-Interference (ISI) in this scenario. But, such a perfect ISI cancellation is not achievable even at infinite SNR when both L and M are finite. However, LMSE filter will be the same as that of zero-forcing filter at infinite SNR irrespective of L and M.

5)

Given that

$$U(z) = \frac{0.36}{1 - 0.8z^{-1}}\, I(z) \qquad \text{and} \qquad D(z) = (1 + 2z^{-1})\, U(z) + V(z)$$

Hence,

$$u(n) = 0.36 \sum_{p=0}^{\infty} (0.8)^p\, i(n-p) \qquad \text{and} \qquad d(n) = u(n) + 2\, u(n-1) + v(n)$$

(a)

$$\begin{aligned}
\mathbb{E}\left[u(n)u(n-l)\right] &= \mathbb{E}\left[\left\{0.36 \sum_{p=0}^{\infty} (0.8)^p\, i(n-p)\right\}\left\{0.36 \sum_{q=0}^{\infty} (0.8)^q\, i(n-l-q)\right\}\right] \\
&= (0.36)^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (0.8)^{p+q}\, \mathbb{E}\big[i(n-p)i(n-l-q)\big] \\
&= (0.36)^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (0.8)^{p+q} \delta(q-p+l) \qquad \left[\ \because \mathbb{E}\left[i(n)i(n-j)\right] = \delta(j)\right] \\
&= (0.36)^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (0.8)^{2p-l} \delta(p-l-q) \\
&= (0.36)^2 \sum_{p=0}^{\infty} (0.8)^{2p-l}\, \mathcal{U}(p-l)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[u^2(n)\right] &= (0.36)^2 \times \frac{1}{1-(0.8)^2} = 0.36 \\
\mathbb{E}\left[u(n)u(n-1)\right] &= (0.36)^2 \times \frac{0.8}{1-(0.8)^2} = 0.36 \times 0.8
\end{aligned}$$

Thus,

$$\boxed{\boldsymbol{R} = 0.36 \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}}$$

$$\begin{aligned}
\mathbb{E}\left[d(n)u(n)\right] &= \mathbb{E}\left[u^2(n)\right] + 2\,\mathbb{E}\left[u(n)u(n-1)\right] + \mathbb{E}\left[u(n)v(n)\right] \\
&= 0.36 + 2 \times 0.36 \times 0.8 + 0 \\
&= 0.936
\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[d(n)u(n-1)\right] &= \mathbb{E}\left[u(n)u(n-1)\right] + 2\,\mathbb{E}\left[u^2(n-1)\right] + \mathbb{E}\left[u(n-1)v(n)\right] \\ &= 0.36 \times 0.8 + 2 \times 0.36 + 0 \\ &= 1.008\end{aligned}$$

Hence,

$$\boxed{\boldsymbol{p} = \begin{pmatrix} 0.936 \\ 1.008 \end{pmatrix}}$$

(b)

$$\boldsymbol{R}^{-1} = \frac{1}{(0.36)^2}\begin{pmatrix} 1 & -0.8 \\ -0.8 & 1 \end{pmatrix}$$

Thus,

$$\boxed{\boldsymbol{w}_{opt} = \boldsymbol{R}^{-1}\boldsymbol{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

(c)

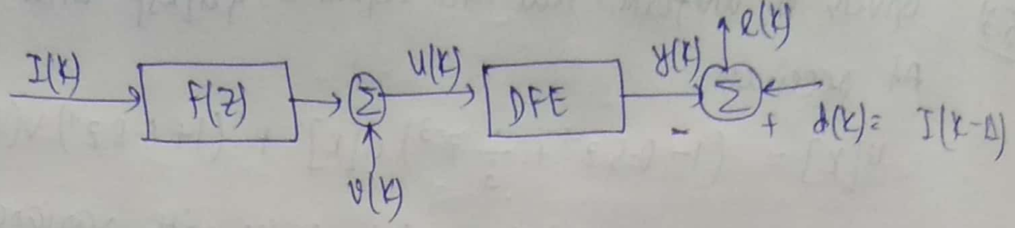
$$\begin{aligned}\mathbb{E}\left[d^2(n)\right] &= \mathbb{E}\left[(u(n) + 2u(n-1) + v(n))^2\right] \\ &= \mathbb{E}\left[u^2(n)\right] + 4\ \mathbb{E}\left[u^2(n-1)\right] + \mathbb{E}\left[v^2(n)\right] \\ &\qquad + 4\ \mathbb{E}\left[u(n)u(n-1)\right] + 2\ \mathbb{E}\left[u(n)v(n)\right] + 4\ \mathbb{E}\left[u(n-1)v(n)\right] \\ &= 0.36 + 4 \times 0.36 + 0.1 + 4 \times 0.36 \times 0.8 + 0 + 0 \\ &= 3.052\end{aligned}$$

Hence,

$$\begin{aligned}J_{min} &= \mathbb{E}\left[d^2(n)\right] - \boldsymbol{p}^H \boldsymbol{w}_{opt} \\ &= 3.052 - 2.952\end{aligned}$$

$$\boxed{J_{min} = 0.1}$$

Q.6)



Given: $F(z) = 1 + 2z^{-1} - z^{-3}$ and additive noise variance, $\sigma_v^2 = 0.1$

$\Delta = 1$, no. of feed forward taps $L = 3$

no. of feed back taps $M = 2$.

Let, $$\underline{a}(k) = \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-L+1) \\ I(k-\Delta-1) \\ \vdots \\ I(k-\Delta-M) \end{bmatrix}$$ and $R = E[\underline{a}(k)\underline{a}^*(k)]$,

$p = E[d(k)\,\underline{a}(k)]$

At receiver,

$$u(k) = \begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} I(k) \\ I(k-1) \\ I(k-2) \\ I(k-3) \end{bmatrix} + v(k)$$

Hence,

$$\underline{a}(k) = \begin{bmatrix} u(k) \\ u(k-1) \\ u(k-2) \\ I(k-2) \\ I(k-3) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} I(k) \\ I(k-1) \\ I(k-2) \\ I(k-3) \\ I(k-4) \\ I(k-5) \end{bmatrix}}_{\underline{I}} + \underbrace{\begin{bmatrix} v(k) \\ v(k-1) \\ v(k-2) \\ v(k-3) \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}}$$

$$R = E\left[(H\underline{I} + \underline{v})(H\underline{I} + \underline{v})^*\right]^*$$

$$= E\left[(H\underline{I} + \underline{v})(\underline{I}^* H^* + \underline{v}^*)\right]$$

$$= E\left[H\underline{I}\,\underline{I}^* H^* + H\underline{I}\,\underline{v}^* + \underline{v}\,\underline{I}^* H^* + \underline{v}\,\underline{v}^*\right]$$

$= HH^* + VV^*$ (cross terms' expectation will be zero) $(E(II^H) = 1)$

$\therefore R = HH^* + \sigma_v^2 I_{5\times5} \rightarrow$ (identity matrix)

$$R = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} + 0.1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 6 & 2 & -2 & 0 & -1 \\ 2 & 6 & 2 & 2 & 0 \\ -2 & 2 & 6 & 1 & 2 \\ 0 & 2 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

$$R = \begin{bmatrix} 6.1 & 2 & -2 & 0 & -1 \\ 2 & 6.1 & 2 & 2 & 0 \\ -2 & 2 & 6.1 & 1 & 2 \\ 0 & 2 & 1 & 1.1 & 0 \\ -1 & 0 & 2 & 0 & 1.1 \end{bmatrix}$$

and, $P = E\left[I(k-1)\left(H\underline{I} + \underline{V}\right)\right] = H\,E\left[I(k-1)\,\underline{I}\right] + 0$

$$= \underline{H} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{since } E\left[I(n)\,I(m)\right] = \begin{cases} 0, & \text{if } n \neq M \\ 1, & \text{if } n = M \end{cases}$$

$$\therefore P = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$