EE4140 Digital Communication Systems

Tutorial #2

1.(a) Average energy, $E_a = \frac{1}{4} \sum E_i = 4J$

where E_i is the energy of the i^{th} point of the constellation.

Constellation: $\{3d, d, -d, -3d\}$

$$E_a = \frac{1}{4}\left[9d^2 + d^2 + d^2 + 9d^2\right] = \underline{\underline{5d^2}}$$

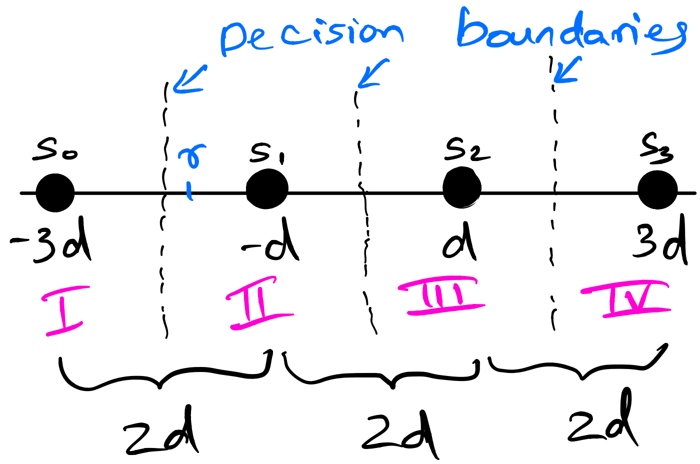
$$5d^2 = 4 \Rightarrow d = \sqrt{\frac{4}{5}}$$

$E_a = 5d^2 = \frac{5}{4}(2d)^2$ where $2d$ is the distance between neighboring points of the constellation.

(b) For $\alpha =$ $\quad z(k) = I(k) + v(k)$

$\sigma^2 = \frac{N_0}{2}$ and $\sigma = \sqrt{\frac{N_0}{2}}$ $\quad v(k) \sim \mathcal{N}(0, \frac{N_0}{2})$

4PAM constellation:



Suppose we have a received signal r that falls in interval II as shown. Error happens when $\Pr(r - s_1 > d)$ or, if r is on the other side of s_1, $\Pr(s_1 - r > d)$. Thus, we evaluate:

$$\Pr(|r - s_1| > d) = 2 \times \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \int_d^\infty e^{-x^2/2 \times \frac{N_0}{2}} dx$$

Let $t = x/\sqrt{N_0/2}$

If $x = d$, $t = d/\sqrt{N_0/2}$. $dt = \dfrac{dx}{\sqrt{N_0/2}}$

$$\therefore \Pr(|r - s_1| > d) = \frac{2}{\sqrt{2\pi}} \int_{d/\sqrt{N_0/2}}^\infty e^{-t^2/2} dt$$

$$= 2\,Q\left(\frac{d}{\sqrt{N_0/2}}\right) = 2q(d)$$

Similarly $P_r(|r - s_2| > d) = 2q(d)$.

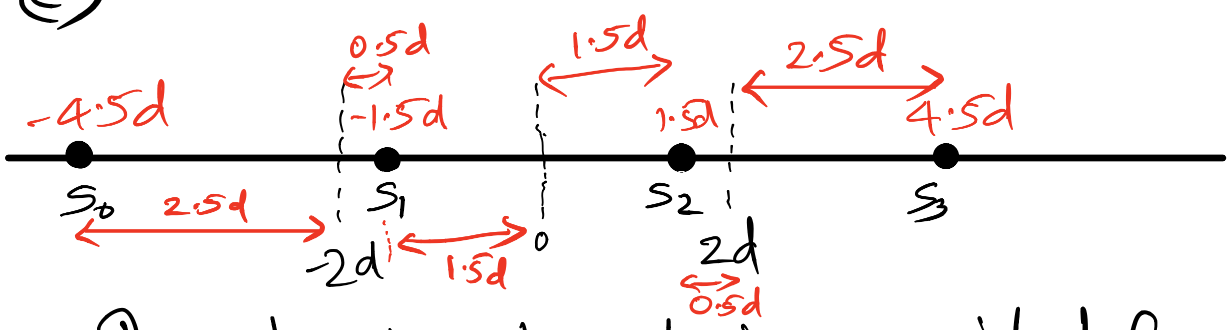
For s_0 & s_3, error only happens in one direction, so probability of error is $q(d)$ each.

Thus average probability of error

$$P_{e_{avg}} = \frac{1}{4}\left(q(d) + 2q(d) + 2q(d) + q(d)\right)$$

$$= \underline{\underline{\frac{3}{2}q(d)}}$$

(c)



The decision boundaries are ideal for $\alpha = 1$. But since $\alpha = 1.5$, the symbol positions scale by 1.5.

For s_0 & s_3, error occurs only in one direction, when received symbol is further than $2.5d$. But for s_1 and s_2, error can occur in both directions, at distances $0.5d$ & $1.5d$.

$$P_{c,s_0} = P_{c,s_3} = 1 - q(2.5d) \Rightarrow P_{e_0} = P_{e_3} = q(2.5d)$$

$$P_{e_1} = P_{e_2} = q(1.5d) + q(0.5d).$$

$$\therefore P_{e,avg} = \frac{1}{4}\left[2q(2.5d) + 2q(1.5d) + 2q(0.5d)\right]$$

$$= \frac{1}{2}\left[q(2.5d) + q(1.5d) + q(0.5d)\right]$$

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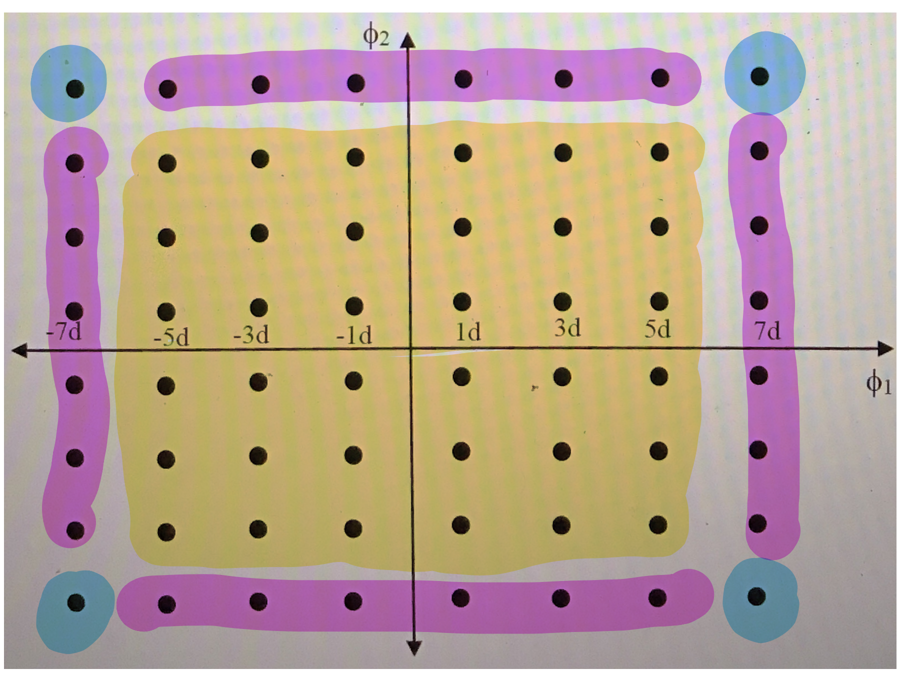
2. $P_e = 1 - P_c$



Following the logic used in the lecture for 16 QAM, we have 4 points (shaded in blue) that have $P_c = (1-q)^2$ for both inphase and quadrature components, 24 points where $P_c = (1-q)(1-2q)$ and the remaining 36 points where $P_c = (1-2q)^2$.

Thus, $P_{e,avg} = 1 - P_{c,avg}$

$$= 1 - \frac{1}{64}\left[4(1-q)^2 + 24(1-q)(1-2q) + 36(1-2q)^2\right]$$

$$= 1 - \frac{1}{16}\left[(1-q)^2 + 6(1-q)(1-2q) + 9(1-2q)^2\right]$$

$$= 1 - \frac{1}{16}\left[1-2q+q^2 + 6(1-3q+2q^2) + 9(1-4q+4q^2)\right]$$

$$= 1 - \frac{1}{16}\left[16 - 56q + 49q^2\right]$$

$$= \frac{1}{16}\left[56q - 49q^2\right] \quad \text{— (1)}$$

(a) If we consider the nearest neighbor based union bound, then the blue-shaded symbols have two nearest neighbors at distance 2d, The pink-shaded symbols have 3 nearest neighbors and the yellow shaded symbols have 4 nearest neighbors. $q(d)$ is the error corresponding to each. Then,

$$P_{UB} = \frac{1}{64}\left[2 \times (4q) + 3 \times (24q) + 4 \times (36q)\right]$$

$$= \frac{1}{16}\left[56\,q(d)\right] \quad -(2)$$

(b) Comparing (1) & (2),

$$P_{e,avg} = P_{UB} - \frac{49}{16}q^2$$

Numerical evaluation may be done by substituting in the q function.

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3. $s(t) = I_1(k)g(t)\cos 2\pi f_c t + I_2(k)g(t)\sin 2\pi f_c t$

$$g(t) = \sqrt{\frac{2}{T}}$$

Assuming $\frac{1}{f_c}$ is an integral multiple of T,

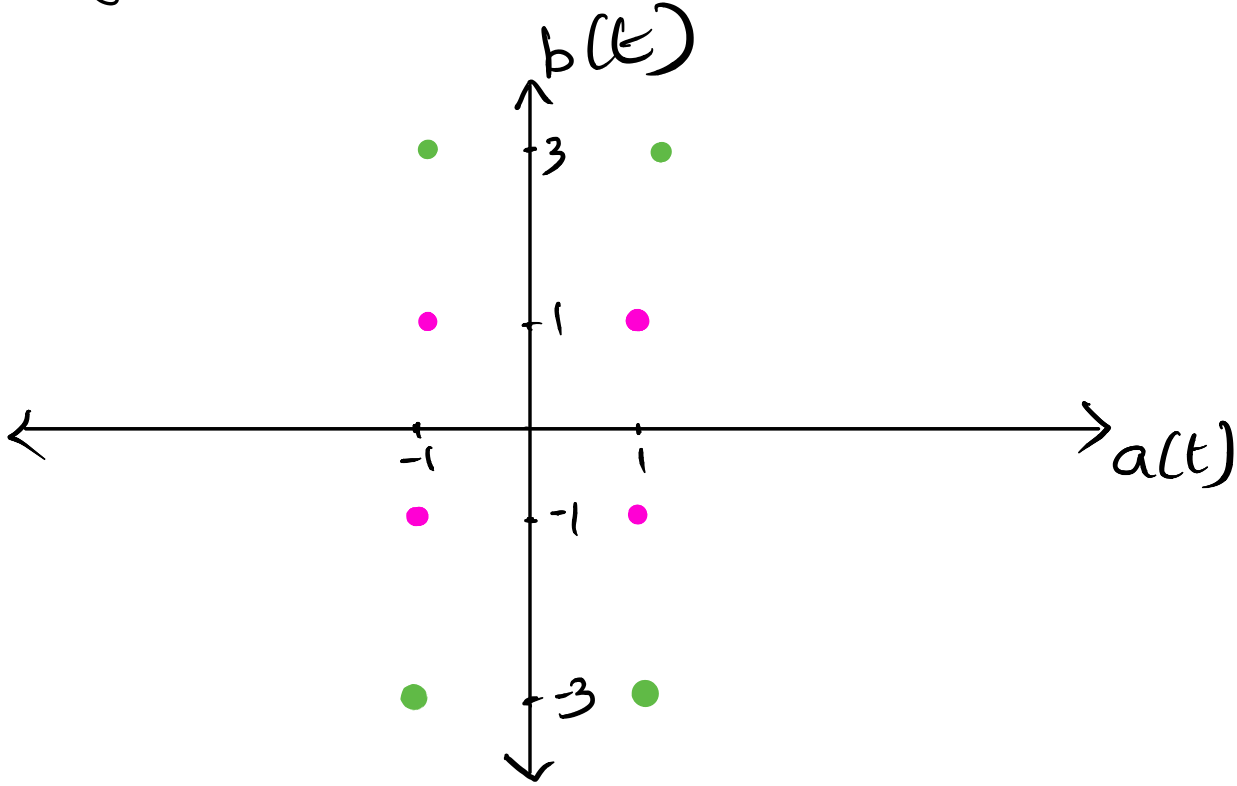
$$\left[\int_0^T \left|\sqrt{\frac{2}{T}}\cos 2\pi f_c t\right|^2 dt\right]^{1/2} = 1, \quad \left[\int_0^T \left|\sqrt{\frac{2}{T}}\sin 2\pi f_c t\right|^2 dt\right]^{1/2} = 1$$

Also the cos and sin terms are orthogonal to each other. Thus,

$$a(t) = \sqrt{\frac{2}{T}}\cos 2\pi f_c t \quad \text{and} \quad b(t) = \sqrt{\frac{2}{T}}\sin 2\pi f_c t$$

form an orthonormal basis set.

Signal Constellation:



(b) $E_a = \frac{1}{8}\left(4(3^2+1^2) + 4(1^2+1^2)\right)$

$$= \frac{1}{2}(12) = 6$$

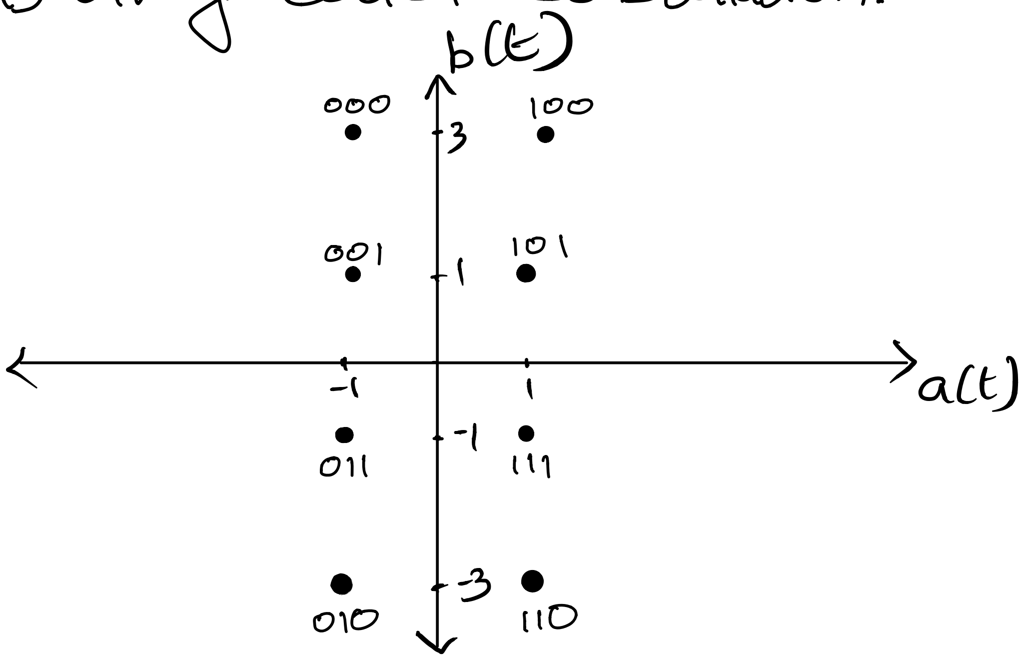
(c) The exact expression can be found in a manner similar to the 64 QAM example.

$$P_c = \frac{1}{8}\left[4(1-q)^2 + 4(1-2q)(1-q)\right]$$

$$= \frac{1}{2}\left[3q^2 - 5q + 2\right]$$

$$P_{e,avg} = \frac{1}{2}\left[5q - 3q^2\right]$$

(d) Gray Coded Constellation:



We need to ensure that adjascent symbols do not differ by more than a bit.

The bit error probability is approximately

$$P_b = \frac{P_a}{3} \quad \text{for this constellation}$$

We have 3 bits and the most likely error is crossing over to the nearest symbols. Since we only have 1 bit difference to the nearest symbol,

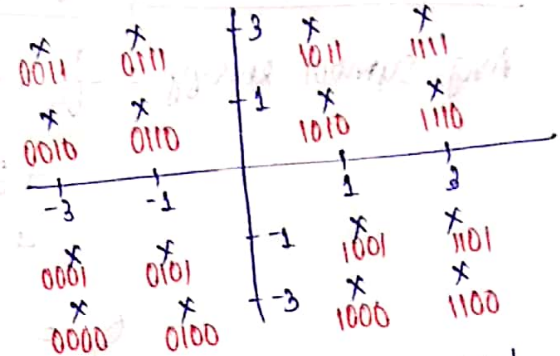
$P_b = P_a/3$.

Q.4) Sol^n: Considering 4-QAM (set x)
16-QAM (set y)
64-QAM (set z)

(a) 4-QAM : 2 bits/symbol
16-QAM : 4 bits/symbol
64-QAM : 6 bits/symbol

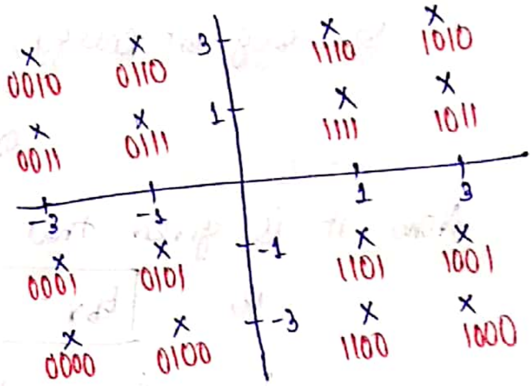
(b) If we consider representing 16-QAM with binary bit representation as follows:

I	bits	Q	bits
-3	00	-3	00
-1	01	-1	01
+1	10	+1	10
+3	11	+3	11

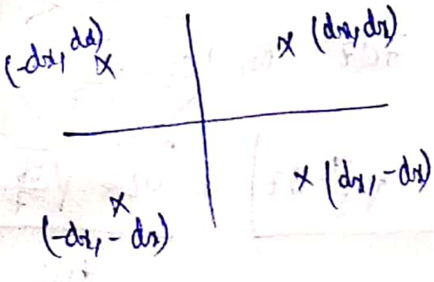


then the above 16-QAM constellation can be represented with Gray coding as follows:

I	bits	Q	bits
-3	00	-3	00
-1	01	-1	01
+1	11	+1	11
+3	10	+3	10



(c) Minimum distance of 4-QAM (d_x):



Avg. symbol energy $= \frac{1}{4}\left[4\times(d_x^2+d_x^2)\right]$

and we have 2 bits/symbol in 4-QAM

so, avg. bit energy,

$$E_{bx} = \frac{1}{2}\times\frac{1}{4}\left[4\times(d_x^2+d_x^2)\right]$$

$$= \frac{1}{2}\times 2d_x^2$$

or, $\boxed{E_{bx} = d_x^2}$ —①—

Minimum distance of 16-QAM (d_y):

Avg. symbol energy $= \frac{1}{16}\left[4\times(d_y^2+d_y^2+9d_y^2+d_y^2+9d_y^2+9d_y^2+d_y^2+9d_y^2)\right]$

$$= \frac{4}{16}\left[2d_y^2+2(10d_y^2)+18d_y^2\right]$$

$= \frac{4}{16} \times 40 d_y^2$

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We have 4 bits/symbol for 16-QAM

so, avg bit energy, $E_{by} = \frac{1}{4} \times \frac{4}{16} \times 40 d_y^2$

or, $\boxed{E_{by} = \frac{5}{2} d_y^2}$ — (2) —

Minimum distance of 64-QAM:

Avg. symbol energy $= \frac{1}{64} \times 4 \left[2d_z^2 + 2(10d_z^2) + 18d_z^2 + 2(26d_z^2) + 2(34d_z^2) + 3(50d_z^2) + 2(58d_z^2) + 2(74d_z^2) + 98d_z^2\right]$

or, $= \frac{4}{64} \times 672\, d_z^2$

For 64-QAM we have 6 bits/symbol.

so, avg. bit energy, $E_{bz} = \frac{1}{6} \times \frac{4}{64} \times 672\, d_z^2$

or, $\boxed{E_{bz} = 7d_z^2}$ — (3) —

Now, it is given that the E_b's are same,

ie, $\boxed{E_{bx} = E_{by} = E_{bz} = E_b}$

$\therefore$ From (1) we have, $E_b = d_x^2$ or, $d_x = \sqrt{E_b}$

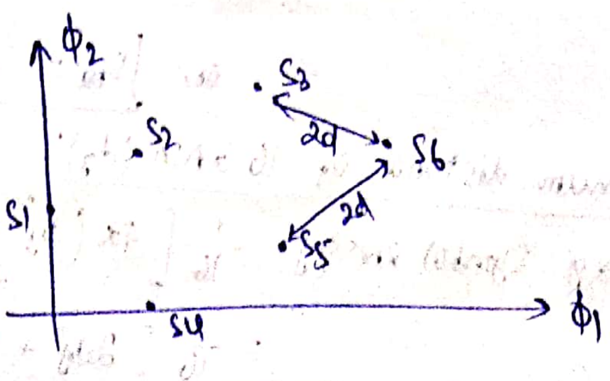
From (2) we have, $E_b = \frac{5}{2} d_y^2$ or, $d_y = \sqrt{\frac{2}{5} E_b}$

& from (3) we have, $E_b = 7d_z^2$ or, $d_z = \sqrt{\frac{E_b}{7}}$

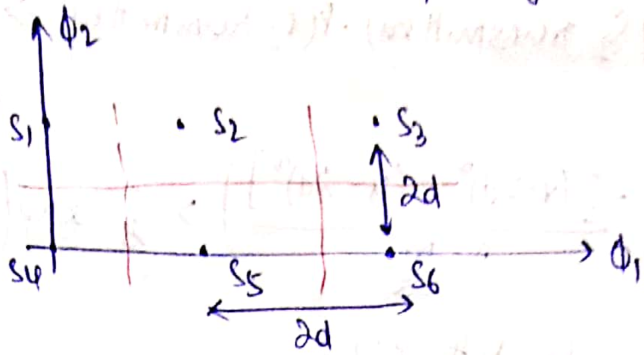
$\therefore$ $\boxed{d_x = \sqrt{7}\, d_z}$, $\boxed{d_y = \sqrt{\frac{14}{5}}\, d_z}$ & $\boxed{d_z = \sqrt{\frac{E_b}{7}}}$

Ans:

Q.5) Soln: Given,



We know, rotation of a constellation doesn't change the P_e.
so, rotating the above constellation, we get:



(a)

Considering the constellation points S_1, S_3, S_4 and S_6, we have

$$P_c = (1-q)(1-q)$$

or, $\boxed{P_c = (1-q)^2}$

and considering the points S_2 and S_5, we have

$$\boxed{P_c = (1-q)(1-2q)}$$

$$\therefore P_c(avg) = \frac{1}{6}\left[(1-q)^2 \times 4 + (1-q)(1-2q)\times 2\right]$$

$$= \frac{1}{6}\left[(1+q^2-2q)\times 4 + (1-3q+2q^2)\cdot 2\right]$$

$$= \frac{1}{6}\left[4+4q^2-8q+2-6q+4q^2\right]$$

$$= \frac{1}{6}\left[6-14q+8q^2\right]$$

$$\& \; P_e(avg) = 1 - P_c(avg) = 1 - \left[1-\frac{14}{6}q+\frac{8}{6}q^2\right]$$

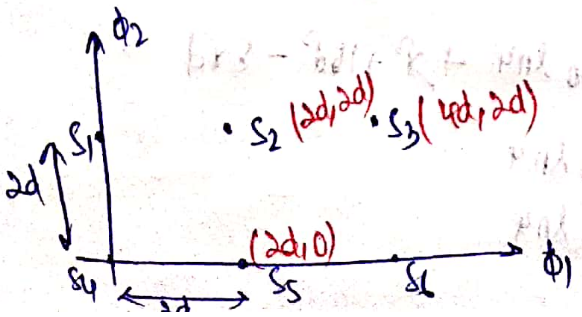
$$= \frac{7}{3}q - \frac{4}{3}q^2$$

or, $\boxed{P_e(avg) = \frac{1}{3}\left[7q-4q^2\right]}$ Ans:

(b) Now given, $P(S_2) = P(S_5) = \frac{1}{3}$ & $P(S_1) = P(S_4) = P(S_3) = P(S_6) = \frac{1}{12}$.

We know from Baye's Rule,

$$P(S_i \text{ transmitted} \mid \text{"S" received}) = P(\text{"S" received} \mid S_i \text{ is transmitted}) \cdot P(S_i \text{ is transmitted})$$



We consider the point S_2 and find its decision boundary with the point S_5 and S_3.

(i) Boundary with S_5:

$P(\text{'s' received} \mid S_2 \text{ transmitted}) \cdot P(S_2 \text{ transmitted}) > P(\text{'S' received} \mid S_5 \text{ transmitted}) \cdot P(S_5 \text{ transmitted})$

or, $$\cancel{\frac{1}{3}} \cdot \exp\left(\frac{-[(x-2d)^2+(y-2d)^2]}{\cancel{2} \cdot \frac{N_0}{\cancel{2}}}\right) > \cancel{\frac{1}{3}} \cdot \exp\left(\frac{-[(x-2d)^2+(y-0)^2]}{\cancel{2} \cdot \frac{N_0}{\cancel{2}}}\right)$$

or, Taking ln both sides,

$$\cancel{-}\left[\cancel{(x-2d)^2}+(y-2d)^2\right] > \cancel{-}\left[\cancel{(x-2d)^2}+y^2\right]$$

or, $$(y-2d)^2 < y^2$$

or, $$\cancel{y^2}+4d^2-4dy < \cancel{y^2}$$

or, $$4d(d-y) < 0$$

or, $$d-y<0$$

or, $$\boxed{y>d}$$ — boundary with S_5

(ii) Boundary with S_3:

$P(\text{'s' received} \mid S_2 \text{ transmitted}) \cdot P(S_2 \text{ transmitted}) > P(\text{'S' received} \mid S_3 \text{ transmitted}) \cdot P(S_3 \text{ transmitted})$

or, $$\frac{1}{3} \exp\left(\frac{-[(x-2d)^2+(y-2d)^2]}{\cancel{2} \cdot \frac{N_0}{\cancel{2}}}\right) > \frac{1}{12} \cdot \exp\left(\frac{-[(x-4d)^2+(y-2d)^2]}{\cancel{2} \cdot \frac{N_0}{\cancel{2}}}\right)$$

or, $$\exp\left(\frac{-[(x-2d)^2+(y-2d)^2]}{N_0}\right) > \frac{1}{4} \exp\left(\frac{-[(x-4d)^2+(y-2d)^2]}{N_0}\right)$$

Taking ln both sides,

$$-\frac{[(x-2d)^2+\cancel{(y-2d)^2}]}{N_0} > -\ln 4 - \frac{[(x-4d)^2+\cancel{(y-2d)^2}]}{N_0}$$

or, $$\frac{(x-2d)^2}{N_0} < \ln 4 + \frac{(x-4d)^2}{N_0}$$

or, $$\cancel{x^2}+4d^2-4xd < N_0 \ln 4 + \cancel{x^2}+16d^2-8xd$$

or, $$4xd - 12d^2 < N_0 \ln 4$$

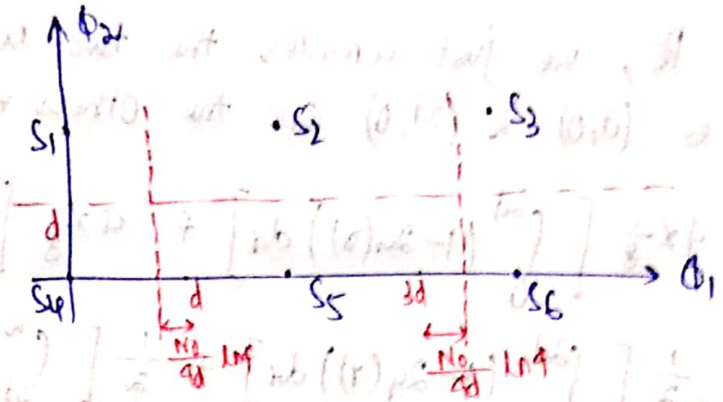
or, $$4d(x-3d) < N_0 \ln 4$$

or, $x - 3d < \frac{N_0 \ln 4}{4d}$

or, $\boxed{x < \frac{N_0 \ln 4}{4d} + 3d}$ → boundary with S_3

Similarly if we consider the boundary of S_2 with S_1 we would get: $\boxed{x > d - \frac{N_0}{4d} \ln 4}$ — boundary with S_1

So, now the constellation diagram with the decision regions could be re-drawn as below.



It verifies the fact that since $P(S_2) > P(S_1)$ & $P(S_2) > P(S_3)$, the decision region is shifted towards that end.

now, the avg. prob. of correctness could be written as:

$$P_c = \frac{4}{12}\left[Q\left(\frac{d}{\sigma}\right) Q\left(\frac{-\frac{N_0}{4d}\ln 4}{\sigma}\right)\right] + \frac{2}{3}\left[Q\left(\frac{d}{\sigma}\right)\left(1 - 2Q\left(\frac{\frac{N_0}{4d}\ln 4}{\sigma}\right)\right)\right]$$

or, $$P_c = \frac{1}{3}\left[Q\left(\frac{d}{\sigma}\right)\left[1 - Q\left(\frac{\sigma}{2d}\ln 4\right)\right]\right] + \frac{2}{3}\left[Q\left(\frac{d}{\sigma}\right)\left(1 - 2Q\left(\frac{\sigma}{2d}\ln 4\right)\right)\right]$$

$\left(\because \sigma^2 = \frac{N_0}{2}\right)$

or, $$P_c = \frac{1}{3}Q\left(\frac{d}{\sigma}\right) - \frac{1}{3}Q\left(\frac{d}{\sigma}\right)Q\left(\frac{\sigma}{2d}\ln 4\right) + \frac{2}{3}Q\left(\frac{d}{\sigma}\right) - \frac{4}{3}Q\left(\frac{d}{\sigma}\right)Q\left(\frac{\sigma}{2d}\ln 4\right)$$

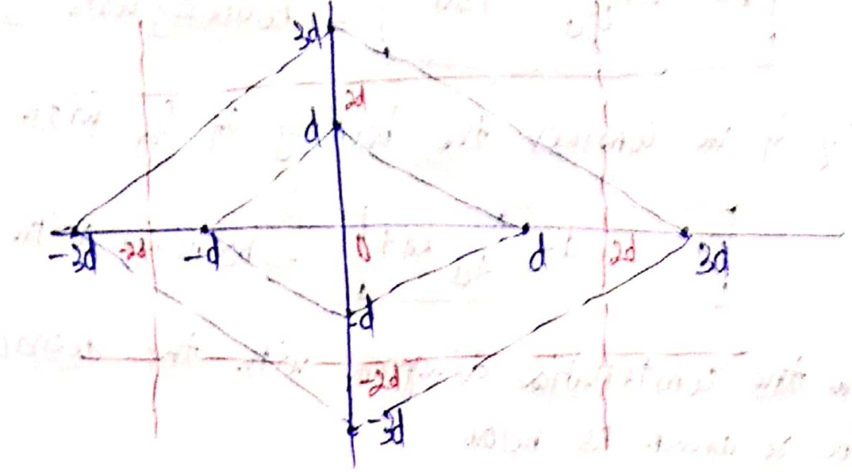
or, $$P_c = Q\left(\frac{d}{\sigma}\right) - \frac{5}{3}Q\left(\frac{d}{\sigma}\right)Q\left(\frac{\sigma}{2d}\ln 4\right)$$

& $P_e = 1 - P_c$

or, $$P_e = 1 - Q\left(\frac{d}{\sigma}\right) + \frac{5}{3}Q\left(\frac{d}{\sigma}\right)Q\left(\frac{\sigma}{2d}\ln 4\right)$$

$\therefore$ or, $$\boxed{P_e = 1 - Q\left(\frac{d}{\sigma}\right)\left[1 - \frac{5}{3}Q\left(\frac{\sigma}{2d}\ln 4\right)\right]}$$ Ans:

Q6) (a) for N=2, the signal constellation would look something like this:



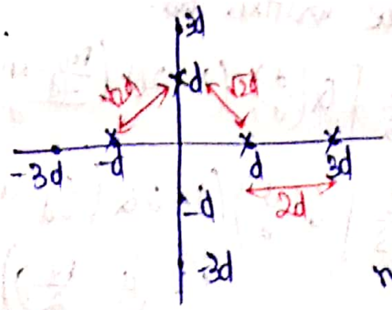
(b) To find P_e, we just consider the two constellation points @ (d,0) & (3d,0) and the others would follow:

$$\therefore P_c = 4 \times \frac{1}{8}\left[\int_0^{2d}(1-2q(x))\,dx\right] + 4\times\frac{1}{8}\left[\int_{2d}^{\infty}(1-q(x))\,dx\right]$$

$$\text{or, } P_c = \frac{1}{2}\left[\int_0^{2d}(1-2q(x))\,dx\right] + \frac{1}{2}\left[\int_{2d}^{\infty}(1-q(x))\,dx\right]$$

where, $q = Q\left(\frac{d}{\sigma}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right)$

(c)



We know for nearest neighbour approx:

$$P_e \simeq \frac{1}{M}\sum_{i=0}^{M-1} n_i\, Q\left(\frac{d_{ni}}{\sqrt{N_0/2}}\right)$$

Consider the point @ (0,d). The nearest neighbours to this point are (d,0) & (-d,0) @ distance $\underline{\sqrt{2}d}$.

$$\therefore \text{ for } (0,d) \text{ point, } P_e \simeq \frac{1}{4}\sum_{i=0}^{3} 2\cdot Q\left(\frac{\sqrt{2}d}{\sqrt{N_0/2}}\right)$$

$$\text{or, } P_e \simeq 2Q\left(\frac{\sqrt{2}d}{\sqrt{N_0/2}}\right)$$

& now considering the point @ (3d,0) the nearest neighbour to this point is @ (d,0) at distance $\underline{2d}$.

$$\therefore \text{ for } (3d,0) \text{ point, } P_e \simeq \frac{1}{4}\sum_{i=0}^{3} 1\cdot Q\left(\frac{2d}{\sqrt{N_0/2}}\right)$$

$$\text{or, } P_e \simeq Q\left(\frac{2d}{\sqrt{N_0/2}}\right)$$

Hence, now the union bound would be, $P_e \leq 2Q\left(\frac{\sqrt{2}d}{\sqrt{N_0/2}}\right) + Q\left(\frac{2d}{\sqrt{N_0/2}}\right)$

or, $\boxed{P_e \leq 2Q\left(\frac{\sqrt{2}d}{\sigma}\right) + Q\left(\frac{2d}{\sigma}\right)}$ Ans: