EE4140: Digital Communication Systems

Tutorial #1.

1) **Compact orthonormal basis set :**

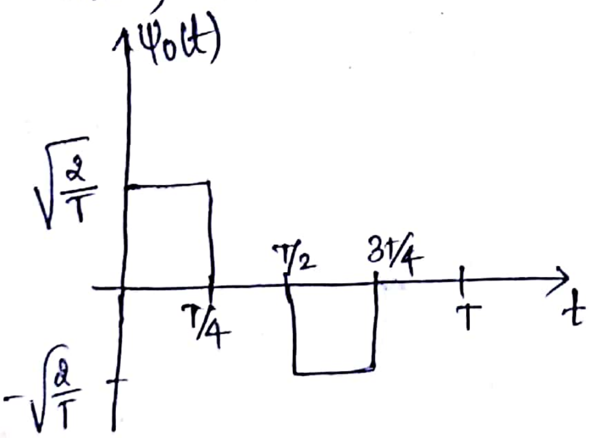
Finding orthonormal basis set by Gram-Schmidt orthogonalization is shown in question ④.

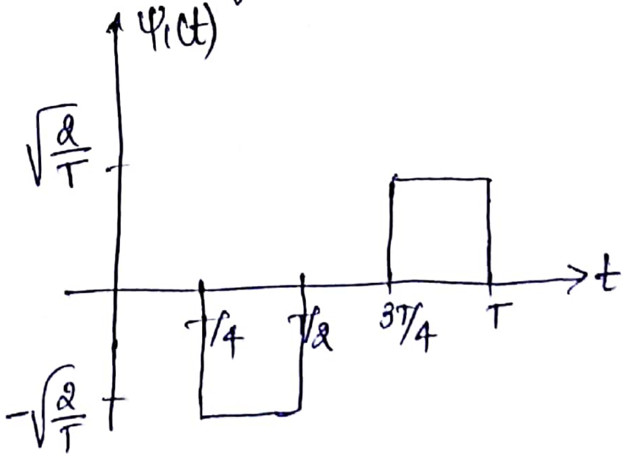
From observation,

$$s_1(t) = s_3(t) + s_6(t)$$
$$s_2(t) = s_3(t) - s_6(t)$$
$$s_4(t) = -(s_3(t) + s_6(t))$$
$$s_5(t) = -(s_3(t) - s_6(t))$$

Hence, all the given signals can be expressed in terms of $s_3(t)$ and $s_6(t)$.

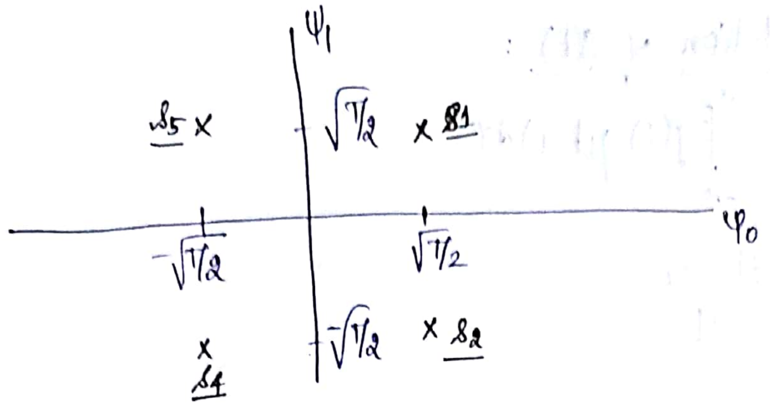
Hence, orthonormal basis set is as follows :





$s_1 = \left(\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}\right)$ $\qquad$ $s_4 = \left(-\sqrt{T/2}, -\sqrt{T/2}\right)$

$s_2 = \left(\sqrt{\frac{T}{2}}, -\sqrt{\frac{T}{2}}\right)$ $\qquad$ $s_5 = \left(-\sqrt{T/2}, \sqrt{T/2}\right)$



Signal Constellation

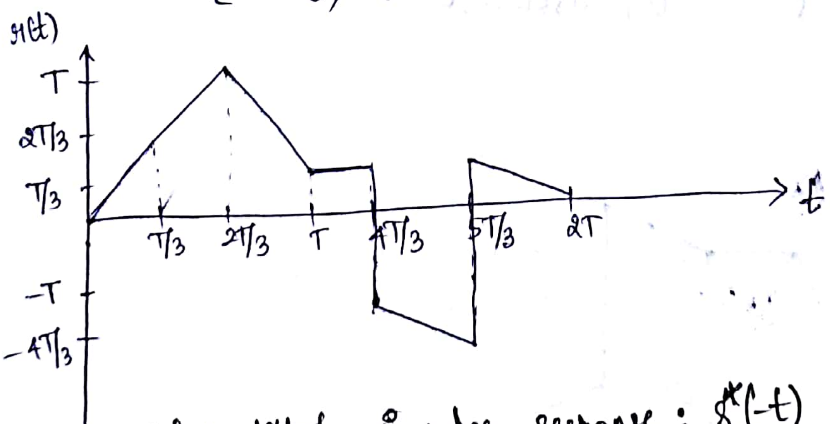
2.) Let $r(t)$ be the received signal without noise.

$$r(t) = g(t) * h(t)$$

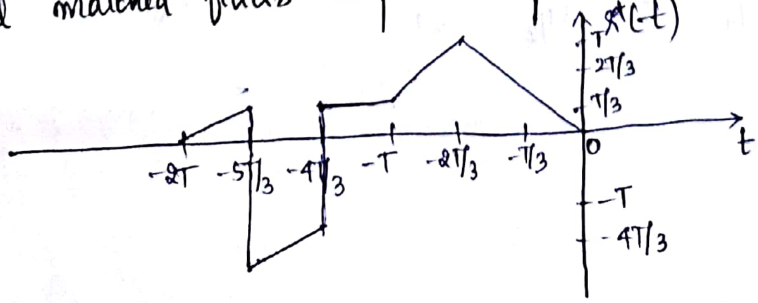
$$r(t) = \int g(\tau)\, h(t-\tau)\, d\tau$$

By performing convolution,

$$r(t) = \begin{cases} 2t, & 0 < t < T/3 \\ \frac{T}{2} + t, & T/3 \le t < 2T/3 \\ \frac{7T}{3} - 2t, & 2T/3 < t < T \\ T/3, & T \le t < 4T/3 \\ T/3 - t, & 4T/3 \le t < 5T/3 \\ 2T - t, & 5T/3 \le t < 2T \\ 0, & \text{otherwise} \end{cases}$$



Ideal matched filter's impulse response : $r^*(-t)$



3) Autocorrelation of $g(t)$:

$$s(t) = \int_{-\infty}^{\infty} g(\tau)\, g(t-\tau)\, d\tau$$

Case 1 : $0 \leq t < T/2$

$$\int_0^t 4\,dt = \underline{4t}$$

Case 2 : $T/2 \leq t < T$

$$\int_{T/2}^{t} 2\,dt + \int_{t-T/2}^{T/2} 4\,dt + \int_0^{t-T/2} 2\,dt = \underline{2T}$$

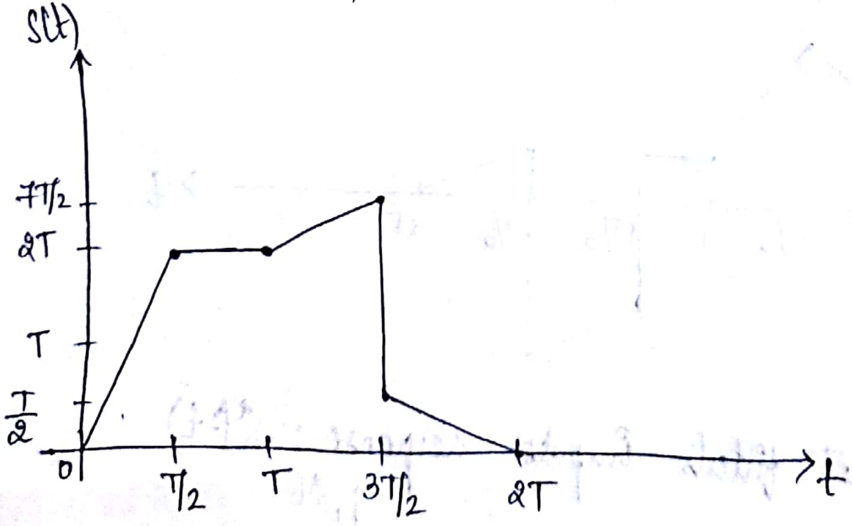
Case 3 : $T \leq t < 3T/2$

$$\int_{t-T}^{T/2} 2\,dt + \int_{T/2}^{t-T/2} 1\cdot dt + \int_{t-T/2}^{T} 2\,dt = \underline{5T-3t}$$

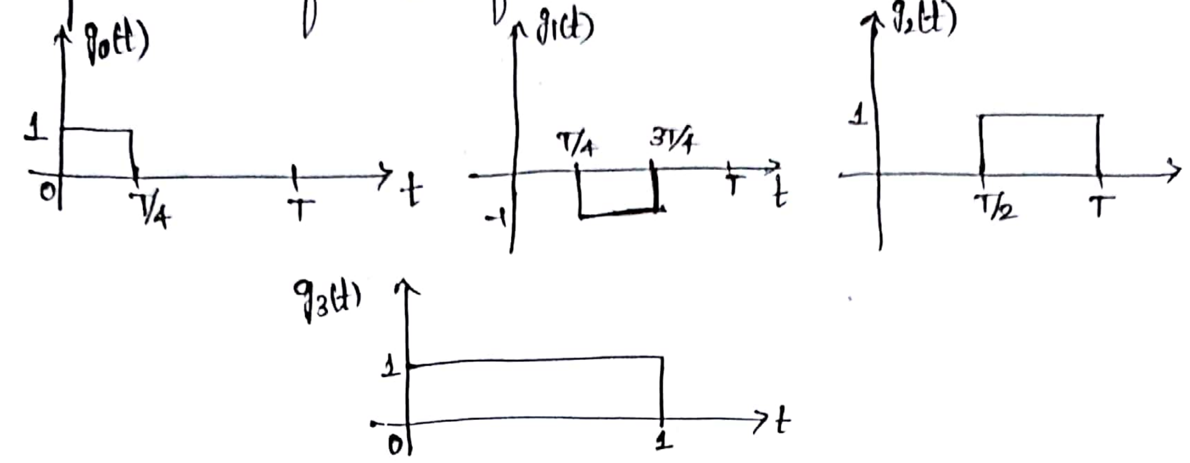
Case 4 : $3T/2 \leq t < 2T$

$$\int_{t-T}^{T} 1\cdot dt = \underline{2T-t}$$

$$s(t) = \begin{cases} 4t, & 0 \leq t < T/2 \\ 2T, & T/2 \leq t < T \\ 5T-3T, & T \leq t < 3T/2 \\ 2T-t, & 3T/2 \leq t < 2T \\ 0, & \text{elsewhere} \end{cases}$$



4) Compact basis functions for:



Method 1:

(i) $d_0(t) \triangleq g_0(t) \Rightarrow \varphi_0(t) = \dfrac{d_0(t)}{\sqrt{\int d_0^2(t)\,dt}} = \dfrac{2}{\sqrt{T}}\, g_0(t)$

(ii) $d_1(t) = g_1(t) - C_{10}\varphi_0(t)$

$C_{10} = \int g_1(t)\varphi_0(t)\,dt = 0$

$\Rightarrow d_1(t) = g_1(t) \Rightarrow \varphi_1(t) = \dfrac{d_1(t)}{\sqrt{\int d_1^2(t)\,dt}} = \sqrt{\dfrac{2}{T}}\, g_1(t)$

(iii) For simplicity let. $d_2(t) = g_3(t) - C_{30}\varphi_0(t) - C_{31}\varphi_1(t)$

$C_{30} = \int g_3(t)\varphi_0(t)\,dt = \int_0^{T/4} \frac{2}{\sqrt{T}}\,dt = \boxed{\frac{\sqrt{T}}{2}}$

$C_{31} = \int g_3(t)\varphi_1(t)\,dt = \int_{T/4}^{3T/4} -\left(\sqrt{2/T}\right)dt = \boxed{-\sqrt{T/2}}$

$$d_2(t) = \begin{cases} 1, & 3T/4 \le t < T \\ 0, & \text{elsewhere} \end{cases} \Rightarrow \varphi_2(t) = \begin{cases} 2/\sqrt{T}, & \frac{3T}{4} \le t < T \\ 0, & \text{elsewhere} \end{cases}$$

(iv) $d_3(t) = g_2(t) - C_{20}\varphi_0(t) - C_{21}\varphi_1(t) - C_{22}\varphi_2(t)$

$C_{20} = \int g_2(t)\varphi_0(t)\,dt = 0$.

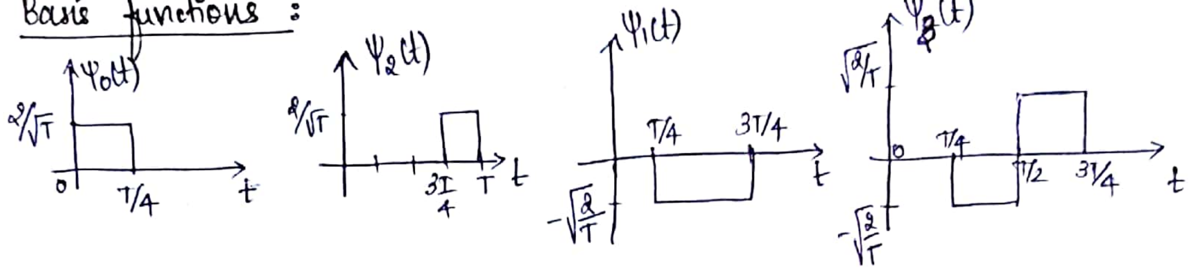
$C_{21} = \int g_2(t)\varphi_1(t)\,dt = \int_{T/2}^{3T/4} \left(-\sqrt{\frac{2}{T}}\right) dt = \boxed{-\frac{1}{2}\sqrt{\frac{T}{2}}}$

$C_{22} = \int g_2(t)\varphi_2(t)\,dt = \int_{3T/4}^{T} \frac{2}{\sqrt{T}}\,dt = \boxed{\frac{\sqrt{T}}{2}}$

$$d_3(t) = g_3(t) + \frac{1}{2} g_1(t) - d_2(t) = \begin{cases} -1/2 , & T/4 \le t < T/2 \\ 1/2 , & T/2 \le t < 3T/4 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \psi_3(t) = \frac{d_3(t)}{\sqrt{\int d_3^2(t)\,dt}} = \begin{cases} -\frac{\sqrt{2}}{\sqrt{T}} , & T/4 \le t \le T/2 \\ +\frac{\sqrt{2}}{\sqrt{T}} , & T/2 \le t \le 3T/4 \\ 0 , & \text{otherwise} \end{cases}$$

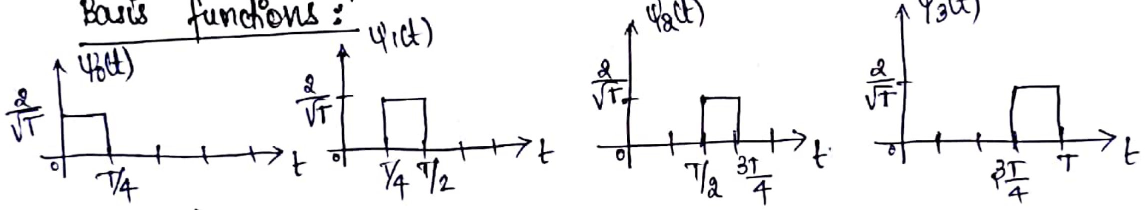
Basis functions :



Method 2 :

In simpler way :

Basis functions :



5) a) Compact orthonormal basis set :

From observation,

$s_4(t) = -s_1(t)$

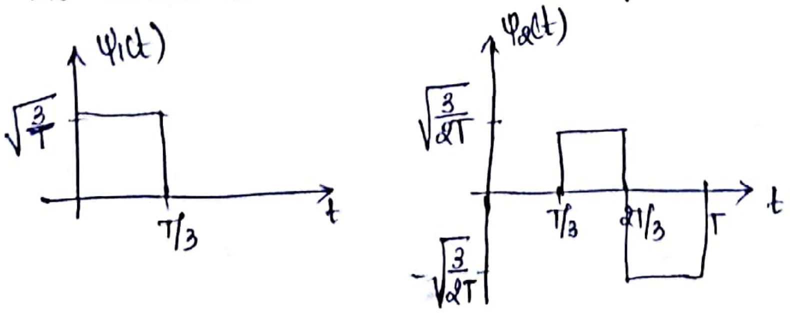
$s_5(t) = -s_2(t)$

$s_6(t) = -s_3(t)$

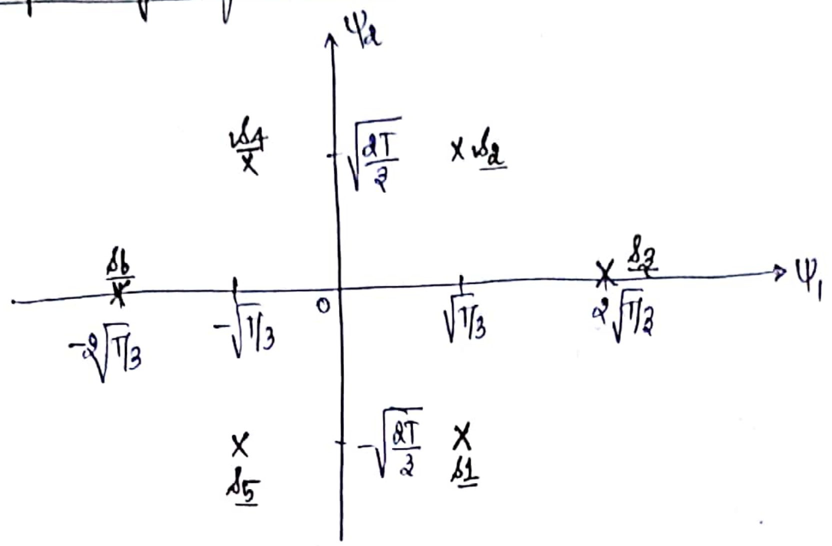
$s_3(t) = s_1(t) + s_2(t)$

$\Rightarrow$ All signals can be expressed only in terms of $s_1(t)$ and $s_2(t)$.

$\therefore$ The orthonormal basis set B as follows :



b) Corresponding signal constellation :



c) Average energy of the constellation

$$E_a = \frac{1}{M}\sum_{i=1}^{M}\|s_i\|^2 = \frac{2}{6}\sum_{i=1}^{3}\|s_i\|^2 = \frac{1}{3}\left(2T + \frac{4T}{2}\right)$$

$$\boxed{E_a = \frac{10}{9}T} \Rightarrow \boxed{T = \frac{9E_a}{10}}$$

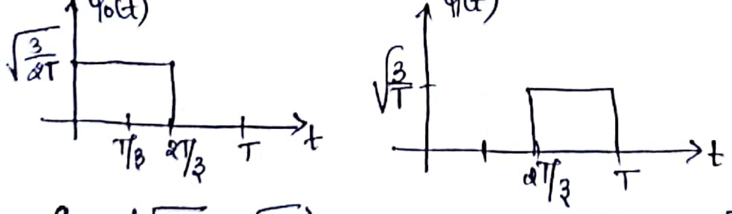
$d_{12} = (2\sqrt{2})\sqrt{\frac{T}{2}}$	$d_{13} = (\sqrt{3})(\sqrt{T/2})$	$d_{14} = (2\sqrt{3})\sqrt{T/3}$	$d_{15} = 2\sqrt{\frac{T}{2}}$
$d_{16} = (\sqrt{11})\sqrt{T/2}$	$d_{23} = (\sqrt{3})(\sqrt{T/2})$	$d_{24} = 2\sqrt{\frac{T}{2}}$	$d_{25} = (2\sqrt{3})\sqrt{\frac{T}{2}}$
$d_{26} = (\sqrt{11})\sqrt{T/3}$	$d_{34} = (\sqrt{11})(\sqrt{T/3})$	$d_{35} = \sqrt{11}\sqrt{\frac{T}{2}}$	$d_{36} = 4\sqrt{\frac{T}{2}}$
$d_{45} = (2\sqrt{2})\sqrt{T/2}$	$d_{46} = (\sqrt{3})(\sqrt{T/2})$	$d_{56} = (\sqrt{3})\sqrt{T/3}$	

Minimum distance :

$$d_{min} = \sqrt{2}\sqrt{\frac{T}{2}} = \sqrt{T} = \boxed{3\sqrt{\frac{E_a}{10}}}$$

6) Minimum distance and multiplicity :

Basis functions



$s_0 = \left(\sqrt{\frac{2T}{3}}, \sqrt{\frac{T}{2}}\right)$; $s_1 = \left(-\sqrt{\frac{2T}{3}}, -2\sqrt{T/2}\right)$;

$s_2 = \left(0, \sqrt{\frac{T}{2}}\right)$; $s_3 = \left(\sqrt{\frac{2T}{3}}, 0\right)$

$d_{01} = (\sqrt{17})\sqrt{\pi/2}$

$d_{12} = (\sqrt{11})\sqrt{\pi/3}$

$d_{02} = \sqrt{2}\sqrt{\frac{T}{2}}$

$d_{13} = (2\sqrt{3})\sqrt{\pi/3}$

$d_{03} = \sqrt{\frac{T}{2}}$

$d_{23} = \sqrt{3}\sqrt{\frac{T}{2}}$

$\boxed{\text{Minimum distance } d_{min} = \sqrt{\pi/2}}$ //

$\boxed{\text{Multiplicity} = 1}$ //