EE4140 -- Lecture #25 Nov.18, 2020

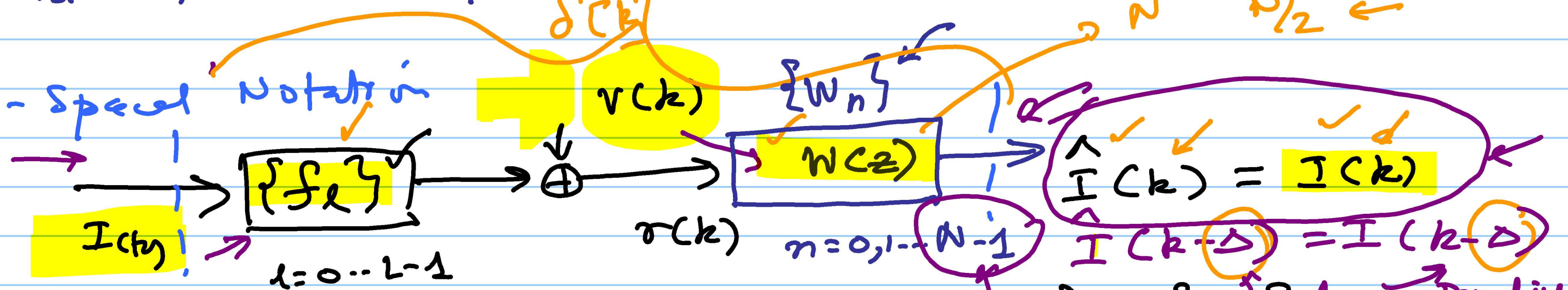
Note Title 09-11-2020

1) Zero Forcing Equalisation

2) Linear MMSE Equalisation

3) Decision Feedback Equalisation ((LMMSE) ⇐ non-linear Structure

1) Symbol-Spaced Notation



→ If $\hat{I}(k) = I(k)$ is desired, how to fix $\{w_n\}$?

FIR → $\{f_\ell\}_L * \{w_\ell\}_N = \delta[k]$ ← Kronecker delta

$L+N-1$

$|S| = 0$

$L+N-2$

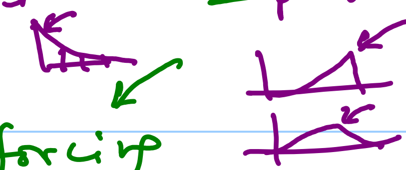
$\Delta \approx \frac{L+N}{2}$

$F(z) \cdot W(z) = 1$

$\Rightarrow \boxed{W(z) = 1/F(z)}$ → zero-forcing

IIR, poles < u.c.

$D=0 \Rightarrow F(z) \to$ min-phase

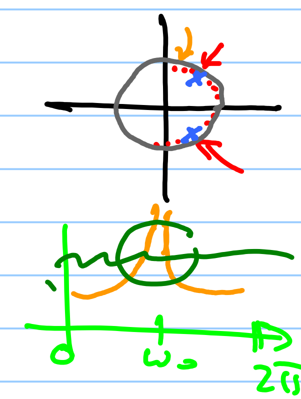


$\{f_\ell\} \to \{f_0, f_1, f_2\}$

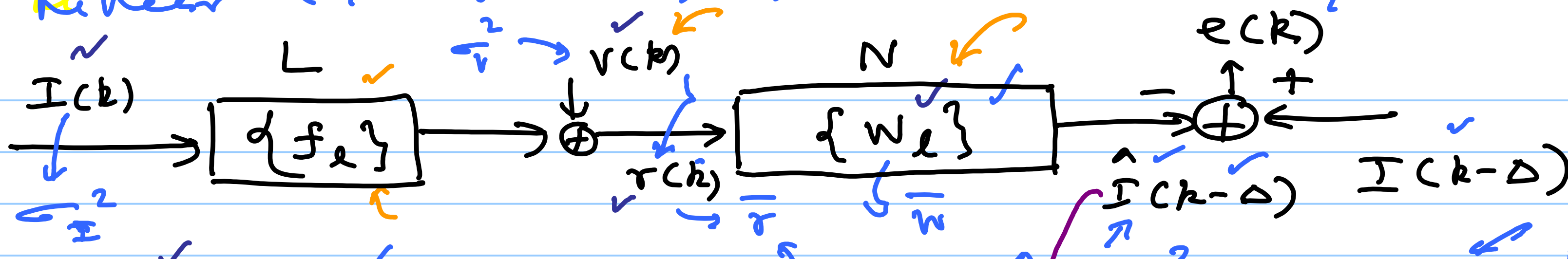
$\{w_\ell\} \to \{w_0, w_1, w_2, \cancel{w_3}\}$

$$\begin{bmatrix} f_0 & 0 & 0 \\ f_1 & f_0 & 0 \\ f_2 & f_1 & f_0 \\ 0 & f_2 & f_1 \\ 0 & 0 & f_2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

5×3   3×1



Linear Equalizer $\longrightarrow$ L-MMSE criterion / Wiener Soln:



$$\min_{\bar{w}} E\left[e^2(k)\right] = E\left[\left(I(k-\Delta) - \hat{I}(k-\Delta)\right)^2\right] = J(\Delta, N, \sigma_v^2)$$

Mean Square Error

$$\bar{r}(k) = \begin{bmatrix} r(k) \\ r(k-1) \\ \vdots \\ r(k-N+1) \end{bmatrix}_{N\times 1}; \qquad \bar{r}^T(k)\,\bar{w} \qquad \bar{w} = \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix}_{N\times 1};$$

$$\min_{\bar{w}} J \Rightarrow \frac{dJ}{d\bar{w}} = 0 \Rightarrow -2\,E\left[\left(I(k-\Delta) - \hat{I}(k-\Delta)\right)\frac{d}{d\bar{w}}\underbrace{\hat{I}(k-\Delta)}_{\bar{r}^T\bar{w}}\right]$$

$T \to *$

take the transpose on both sides.

$$E\left[I(k-\Delta)\, \bar{r}^{T} \right] = E\left[\hat{I}(k-\Delta)\, \bar{r}^{T} \right]$$

$$\underbrace{E\left[\bar{r}\; I(k-\Delta) \right]}_{\triangleq\, \bar{p}_{N\times 1}} = E\left[\bar{r}\; \underbrace{\hat{I}(k-\Delta)}_{\bar{r}^{T}\bar{w}} \right]$$

$\bar{p}$: cross-correlation vector

$$= \underbrace{E\left[\bar{r}\,\bar{r}^{T} \right]}_{\triangleq\, R_{N\times N}} \cdot \bar{w}$$

R: Auto correlation matrix $N\times N$

$$R = E\left[\bar{r}\,\bar{r}^{T} \right] = E\left[\begin{bmatrix} r(k) \\ r(k-1) \\ \vdots \\ r(k-N+1) \end{bmatrix} \begin{bmatrix} r(k) & \cdots & r(k-N+1) \end{bmatrix} \right]$$

$$R = \begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix}_{N\times N} + \sigma_v^2$$

$$R\bar{w} = \bar{p}$$

$$\Rightarrow \boxed{\bar{w}_{opt} = R^{-1}\bar{p}}$$

compromise between noise minimization & ISI minimization.

Example: $L = 3$ $\{f_0, f_1, f_2\}$ $N = 2$ $\{w_0, w_1\}$

$$\bar{\sigma} = \begin{bmatrix} \sigma(k) \\ \sigma(k-1) \end{bmatrix}$$

$$\bar{W}_{opt} = R^{-1}\bar{p} \qquad R = E\left[\bar{\sigma}\,\bar{\sigma}^T\right]$$

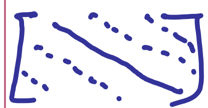
$\Delta = ?$

$$= \begin{bmatrix} E[\sigma^2(k)] & E[r(k)\,\sigma(k-1)] \\ E[\sigma(k-1)\,\sigma(k)] & E[\sigma^2(k-1)] \end{bmatrix};$$

$$\bar{p} = E\left[I(k-\Delta) \begin{bmatrix} r(k) \\ r(k-1) \end{bmatrix} \right]$$

✓ $\sigma(k) = f_0 I(k) + f_1 I(k-1) + f_2 I(k-2) + v(k)$

✓ $\sigma(k-1) = f_0 I(k-1) + f_1 I(k-2) + f_2 I(k-3) + v(k-1)$

 $\longrightarrow R = \begin{bmatrix} (f_0^2+f_1^2+f_2^2)\sigma_I^2 + \sigma_v^2 & (f_0f_1+f_1f_2)\sigma_I^2 \\ (f_0f_1+f_1f_2)\sigma_I^2 & (\cdot\ \cdot\ \cdot)\sigma_I^2+\sigma_v^2 \end{bmatrix};$

$A = A^T$

Symmetric, Toeplitz

✓ $\bar{p}_{\Delta=0} = E\left[I(k-\Delta) \begin{bmatrix} r(k) \\ r(k-1) \end{bmatrix} \right] \xrightarrow{\Delta=0} = \begin{bmatrix} f_0\sigma_I^2 \\ 0 \end{bmatrix};$

✓ $\bar{p}_{\Delta=1} = E\left[I(k-1) \begin{bmatrix} r(k) \\ r(k-1) \end{bmatrix} \right] = \begin{bmatrix} f_1\sigma_I^2 \\ f_0\sigma_I^2 \end{bmatrix};$

$\bar{w}_{opt} = R^{-1}\bar{p}$

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$p^T R^{-1} \bar{p}$ $\quad (R^{-1})^T = R^{-1}$

Exercise: Show that

$$\boxed{J_{min} \triangleq J(\bar{w} = \bar{w}_{opt}) = \sigma_I^2 - \underbrace{\bar{w}_{opt}^T \bar{p}}_{p^T R^{-1} \bar{p}}}$$

For a given $F(z)$, $\sigma_v^2 \longrightarrow J_{min}(\Delta, N)$