Department of Electrical Engineering, IIT Madras

EE3005: Communication Systems

Tutorial #5

1. From the enclosed scanned problem sheets below(<u>*Courtesy*</u>: taken from the book "Probability, RVs, and Stochastic Processes", by A. Papoulis, 2nd Ed., Chapter 6, pp-147,) do the following problems: **6-1** to **6-3**, **6-5** to **6-10**. Use the method of "dummy variable" wherever it makes sense.

PROBLEMS

6-1 If x and y are the zero-one RVs associated with the events $\mathscr{A}$ and $\mathscr{B}$ respectively, (*a*) find the probability masses in the *x-y* plane and (*b*) show that the RVs x and y are independent iff the events $\mathscr{A}$ and $\mathscr{B}$ are independent.

6-2 The RVs x and y are independent and $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Find $f_y(y)$ if

$$f_x(x) = ce^{-cx}U(x) \qquad f_z(z) = c^2 ze^{-cz}U(z)$$

6-3 The RVs x and y are independent and y is uniform in the interval (0, 1). Show that, if $\mathbf{z} = \mathbf{x} + \mathbf{y}$, then

$$f_z(z) = F_x(z) - F_x(z-1)$$

6-4 (*a*) The function $g(x)$ is monotone increasing and $\mathbf{y} = g(\mathbf{x})$. Show that

$$F_{xy}(x, y) = \begin{cases} F_x(x) & \text{if} \quad y > g(x) \\ F_y(y) & \text{if} \quad y < g(x) \end{cases}$$

(*b*) Find $F_{xy}(x, y)$ if $g(x)$ is monotone decreasing.

6-5 Express $F_{zw}(z, w)$ in terms of $f_{xy}(x, y)$ if $\mathbf{z} = \max(\mathbf{x}, \mathbf{y})$, $\mathbf{w} = \min(\mathbf{x}, \mathbf{y})$.

6-6 The RVs x and y are $N(0, 2)$ and independent. Find $f_z(z)$ and $F_z(z)$ if (*a*) $\mathbf{z} = 2\mathbf{x} + 3\mathbf{y}$, and (*b*) $\mathbf{z} = \mathbf{x}/\mathbf{y}$.

6-7 The RVs x and y are independent with

$$f_x(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}U(x) \qquad f_y(y) = \begin{cases} 1/\pi\sqrt{1-y^2} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$$

Show that the RV $\mathbf{z} = \mathbf{xy}$ is $N(0, \alpha)$.

6-8 The RVs x and y are independent with Rayleigh densities

$$f_x(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}U(x) \qquad f_y(y) = \frac{y}{\beta^2} e^{-y^2/2\beta^2}U(y)$$

(*a*) Show that if $\mathbf{z} = \mathbf{x}/\mathbf{y}$, then

$$f_z(z) = \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} U(z) \qquad \text{(i)}$$

(*b*) Using (i), show that for any $k > 0$

$$P\{\mathbf{x} \le k\mathbf{y}\} = \frac{k^2}{k^2 + \alpha^2/\beta^2}$$

6-9 The RVs x and y are independent with exponential densities

$$f_x(x) = \alpha e^{-\alpha x}U(x) \qquad f_y(y) = \beta e^{-\beta y}U(y)$$

Find the densities of the following RVs:

1. $2\mathbf{x} + \mathbf{y}$ 2. $\mathbf{x} - \mathbf{y}$ 3. $\dfrac{\mathbf{x}}{\mathbf{y}}$ 4. $\max(\mathbf{x}, \mathbf{y})$ 5. $\min(\mathbf{x}, \mathbf{y})$

6-10 The RVs x and y are independent and each is uniform in the interval $(0, a)$. Find the density of the RV $\mathbf{z} = |\mathbf{x} - \mathbf{y}|$.

2. In a fair-coin experiment, we define a random process (RP) $X(t)$ as follows: $X(t) = \sin(\pi t)$ is heads show, and $X(t) = 2t$ if tails show.
(a) Find $E[X(t)]$
(b) Find the one-dimensional (first-order) PDF of $X(t_i)$ for (i) $t_1 = 0.25$; (ii) $t_2 = 0.50$; (iii) $t_3 = 1.0$;

3. The RP $X(t) = e^{At}$ is a family of exponentials based on the RV A with a pdf $f_A(a)$. Express $R_X(t_1, t_2)$ and the first order PDF $f_X(t)$ in terms of $f_A(a)$.

4. The RV β be uniform in the interval (0,*T*). The RP is defined by $X(t) = U(t - \beta)$ where $U(.)$ is the unit-step function. Find the expression(s) for $R_X(t_1, t_2)$.

5. Show that if the RP $V(t)$ has $R_V(t_1, t_2) = g(t_1)\delta(t_1 - t_2)$ and the RP $W(t) = \int_0^t V(\tau)d\tau$ then,

$$E[W^2(t)] = \int_0^t g(\tau)d\tau$$

6. Show that if the RP $X(t)$ has $R_X(t_1, t_2) = g(t_1)\delta(t_1 - t_2)$ and the RP $Y(t) = X(t) * h(t)$ where the symbol "*" represents linear convolution, then,

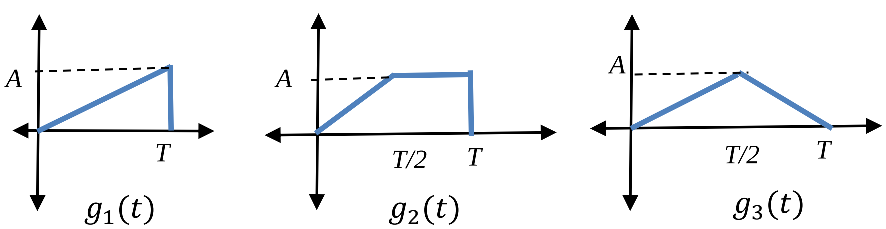
$$E[X(t)Y(t)] = h(0)g(t)$$

7. The discrete RP $X(n)$ is WSS with $R_{XX}(m) = 5\delta(m)$. Given

$$Y(n) - 0.5Y(n-1) = X(n) \qquad (1)$$

(a) Find $E[Y^2(n)], R_{XY}(m_1, m_2), \& \ R_{YY}(m_1, m_2)$ when (1) holds for all n.
(a) Find $E[Y^2(n)], R_{XY}(m_1, m_2), \& \ R_{YY}(m_1, m_2)$ if $Y(-1) = 0$ and (1) holds for all $n \geq 0$.

8. Consider the following "pulse-shapes" and answer the following questions given that θ is a uniform RV between (0,*T*):



(a) If the RP $X(t)$is given by $X(t) = \sum_{k=-\infty}^{+\infty} g_i(t - kT - \theta)$, for each choice of $g_i(t), i = 1,2, \&3$ find (i) The first order PDF $f_X(x)$ and (ii) the expected value $m_X = E[X]$.

(b) If the RP $Y(t)$is given by $Y(t) = \sum_{k=-\infty}^{+\infty} \alpha_k \ g_i(t - kT - \theta)$, where the discrete RV $\alpha_k \in \{-1, +1\}$ takes both values with equal probability, then, for each choice of $g_i(t), i = 1,2, \&3$ find (i) The first order PDF $f_Y(y)$ (ii) the expected value $m_Y = E[Y]$, and (iii) the expression for $R_Y(t_1, t_2)$ and also plot this function if it represents a WSS process.